SOME RESULTS ON π -SOLVABLE AND SUPERSOLVABLE GROUPS

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ABSTRACT. For a finite group G , $\phi_p(G)$, $S_p(G)$, $L(G)$ and $S_p(G)$ are generalizations of the Frattini subgroup of G . We obtain some results on π -solvable, p -solvable and supersolvable groups with the help of the structures of these subgroups.

KEY WORDS AND PHRASES. p -solvable, π -solvable, supersolvable.

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1. INTRODUCTION.

Many authors have considered various generalizations of the Frattini subgroup of a finite group. Deskins [6] considered the subgroup $\phi_p(G)$, Mukherjee and Bhattacharya [4] the subgroup $S_p(G)$ and Bhattacharya [3] the subgroup $L(G)$. In [7], we introduced the subgroup $S_p(G)$ and investigated its influence on solvable group. In this paper, our aim is to prove some results which imply a finite group G to be π -solvable, p -solvable and supersolvable. All groups are assumed to be finite. We use standard notations as found in Gorenstein [8] and denote a maximal subgroup M of G by $M < G$.

2. PRELIMINARIES.

DEFINITION. Let H and K be two normal subgroups of a group G with $K < H$. Then the factor group H/K is called a chief factor of G if there is no normal subgroup N of G such that $K < N < H$, with proper inclusion. Let M be a maximal subgroup of G . Then H is said to be a normal supplement of M in G if $MH = G$. The normal index of M in G is defined as the order of a chief factor H/K , where H is minimal in the set of all normal supplements of M in G and is denoted by $\eta(G : M)$.

(2.1) (Deskins [6, (2.1)], Beidleman and Spencer [2, Lemma-1])

If M is a maximal subgroup of a group G then $\eta(G : M)$ is uniquely determined.

(2.2) (Beidleman and Spencer [2, Lemma-2])

If N is a normal subgroup of a group G and M is a maximal subgroup of G such that $N \subseteq M$ then $\eta(G/N : M/N) = \eta(G : M)$

(2.3) (Mukherjee [9, Theorem-1])

If M is a maximal subgroup of a group G and $M < G$ then $\eta(G : M) = [G : M] = a$ prime.

(2.4) (Baer [1, Lemma-3])

If the group G possesses a maximal subgroup with core 1 then the following properties of G are equivalent.

(1) The indices in G of all the maximal subgroups with core 1 are powers of one and the same prime p .

(2) There exists one and only one minimal normal subgroup of G and there exists a common prime divisor of all the indices in G of all the maximal subgroups with core 1.

(3) There exists a non-trivial solvable normal subgroup of G .

DEFINITION. Let G be a group and p be any prime. The four characteristic subgroups of G , which are analogous to the Frattini subgroup $\phi(G)$, are defined as follows :

$$\begin{aligned} S_p(G) &= \bigcap \{ M : M \in \Sigma_p(G) \} \\ \phi_p(G) &= \bigcap \{ M : M \in \gamma_p(G) \} \\ \Lambda(G) &= \bigcap \{ M : M \in \Lambda(G) \} \\ S_{\mathcal{P}}(G) &= \bigcap \{ M : M \in \Sigma_{\mathcal{P}}(G) \} \end{aligned}$$

where

$$\begin{aligned} \Sigma_p(G) &= \{ M : M < G, [G:M]_p = 1 \text{ and } [G:M] \text{ is composite} \} \\ \gamma_p(G) &= \{ M : M < G, [G:M]_p = 1 \} \\ \Lambda(G) &= \{ M : M < G, [G:M] \text{ is composite} \} \\ \Sigma_{\mathcal{P}}(G) &= \{ M : M < G, \gamma(G:M)_p = 1 \text{ and } \gamma(G:M) \text{ is composite} \} \end{aligned}$$

In case $\Sigma_{\mathcal{P}}(G)$ is empty then we define $G = S_{\mathcal{P}}(G)$ and the same thing is done for the other three subgroups.

(2.5) If H is a subgroup with finite index n in a group G then $\text{core}_G H$ has finite index dividing $n!$

(2.6) (Dutta and Bhattacharyya [7, Theorem-3.5])

If G is p -solvable then $S_{\mathcal{P}}(G)$ is solvable.

DEFINITION. Let M be a maximal subgroup of a group G . Then M is said to be c -maximal if $[G:M]$ is composite.

3. SOME RESULTS ON p -SOLVABLE AND π -SOLVABLE GROUPS.

THEOREM 3.1. Let p be the largest prime dividing $|G|$ and $\Sigma_p(G) \neq \emptyset$. Then G is p -solvable if and only if $\gamma(G:M)_p = [G:M]_p$ for each M in $\Sigma_p(G)$.

PROOF. Let G satisfy the hypothesis of the theorem. Then G is not simple. For, otherwise $|G|_p = \gamma(G:M)_p = [G:M]_p = 1$, where M belongs to $\Sigma_p(G)$, which contradicts the fact that p divides $|G|$. Let N be a minimal normal subgroup of G . If p does not divide $|G/N|$ then G/N is a p' -group and hence it is p -solvable. If p divides $|G/N|$ then p is the largest prime dividing $|G/N|$. If $\Sigma_p(G/N) = \emptyset$ then $G/N = S_p(G/N)$. By Theorem-8(1) [10], $S_p(G/N)$ is solvable and hence G/N is p -solvable. We now assume that $\Sigma_p(G/N) \neq \emptyset$. By Lemma-2 [2], we obtain $\gamma(G/N:M/N)_p = [G/N : M/N]_p$ for each M/N in $\Sigma_p(G/N)$. So by induction, G/N is p -solvable. We note that $S_p(G) \neq G$, since $\Sigma_p(G) \neq \emptyset$. If $N \subseteq S_p(G)$ then N is solvable and so it is p -solvable and consequently G is p -solvable. If $N \not\subseteq S_p(G)$ then there exists M in $\Sigma_p(G)$ such that $N \not\subseteq M$ and so $G = MN$. By hypothesis $|N|_p = \gamma(G:M)_p = [G:M]_p = 1$ and so N is p -solvable and hence G is p -solvable.

The converse follows directly from Theorem 1 [2].

THEOREM 3.2. Let p be the largest prime dividing $|G|$. Then G is p -solvable if the following hold.

- (i) G has a p -solvable c -maximal subgroup M with $\eta(G:M)_p = [G:M]_p$
- (ii) If M_1 and M_2 are c -maximal subgroups of G with $\eta(G:M_1)_p = \eta(G:M_2)_p$ then $[G:M_1]_p = [G:M_2]_p$

REMARK 3.3. The converse of the above theorem is not necessarily true. Let G be a p -group, where p is any prime. Then G is p -solvable, but it has no c -maximal subgroup and so G does not satisfy the hypothesis (i) of the above theorem. If the group G has a c -maximal subgroup then the converse of Theorem 3.2 follows from Theorem 1 [2].

THEOREM 3.4. Let G be a p -solvable group and $\Sigma_{\mathcal{D}}(G) \neq \emptyset$. Then G is π -solvable if and only if $\eta(G:M)_{\pi} = [G:M]_{\pi}$ for each M in $\Sigma_{\mathcal{D}}(G)$.

PROOF. Let the condition of the theorem hold. Let G be simple. Then it immediately follows that either G is a p' -group or is of prime order p . If G is of prime order p then it is solvable and hence π -solvable. If G is a p' -group then $|G|_p = 1$. Also $|G|$ is composite. For, otherwise, G is cyclic and hence it is π -solvable. Let $|G|_{\pi} \neq 1$ and $p_1, p_2, \dots, p_n$ be the set of prime divisors of $|G|$, which belong to π . Let $S(p_i)$ ($i = 1, 2, \dots, n$) denote the Sylow p_i -subgroup of G . Then $S(p_i) \neq G$ for $i = 1, 2, \dots, n$. For, otherwise, G is solvable and hence G is π -solvable. Let M_i be the maximal subgroups of G such that $S(p_i) \subset M_i \subset G$ and so $[G:M_i]_{p_i} = 1$ ($i = 1, 2, \dots, n$). By hypothesis $|G|_{\pi} = \eta(G:M_i)_{\pi} = [G:M_i]_{\pi}$ ($i = 1, 2, \dots, n$). As each $p_i \in \pi$, it follows that $|G|_{\pi} = 1$, a contradiction. So $|G|_{\pi} = 1$ and hence G is π -solvable. We now suppose that G is not simple. Let N be a minimal normal subgroup of G . Then G/N is a p -solvable group. If $\Sigma_{\mathcal{D}}(G/N) = \emptyset$ then $G/N = S_{\mathcal{D}}(G/N)$ and so by (2.6), it follows that G/N is solvable and hence it is π -solvable. We now assume that $\Sigma_{\mathcal{D}}(G/N) \neq \emptyset$. Using Lemma 2 [2], we obtain $\eta(G/N : M/N)_{\pi} = [G/N : M/N]_{\pi}$ for each M/N in $\Sigma_{\mathcal{D}}(G/N)$. By induction, G/N is π -solvable. Let N_1 be another minimal normal subgroup of G . Then G/N_1 is π -solvable. Since $G = G/N \cap N_1$ is isomorphic to a subgroup of the π -solvable group $G/N \times G/N_1$, it follows that G is π -solvable. We may now assume that N is the unique minimal normal subgroup of G . We shall now show that N is π -solvable. We note that $S_{\mathcal{D}}(G) \neq G$, since $\Sigma_{\mathcal{D}}(G) \neq \emptyset$. If $N \in S_{\mathcal{D}}(G)$ then by (2.6) it follows that N is solvable and hence it is π -solvable. If $N \notin S_{\mathcal{D}}(G)$ then there exists M_0 in $\Sigma_{\mathcal{D}}(G)$ such that $N \not\subset M_0$ and so $G = M_0 N$ and $\text{core}_G(M_0) = \langle 1 \rangle$. Let M be any maximal subgroup of G with core 1. Then $N \not\subset M$ and so $G = MN$. Clearly M belongs to $\Sigma_{\mathcal{D}}(G)$. By hypothesis $|N|_{\pi} = \eta(G:M)_{\pi} = [G:M]_{\pi}$. If $|N|_{\pi} = 1$ then N is π -solvable. If $|N|_{\pi} \neq 1$ then there exists a common prime divisor of all the indices in G of all the maximal subgroups with core 1. So by (2.4), N is solvable and hence it is π -solvable. Thus G/N and N are both π -solvable. So G is π -solvable.

The converse follows directly from Theorem 2 [9].

THEOREM 3.5. Let G be a group with $\lambda(G) \neq \emptyset$. Then G is π -solvable if and only if $\eta(G:M)_{\pi} = [G:M]_{\pi}$ for each M in $\lambda(G)$, where $\lambda(G) = \{M : M \triangleleft G, \eta(G:M) \text{ is composite}\}$.

THEOREM 3.6. Let G be a group with $|\lambda(G)| \geq 2$. Then G is π -solvable if and only if $\eta(G:M_1)_{\pi} = \eta(G:M_2)_{\pi}$ implies $[G:M_1]_{\pi} = [G:M_2]_{\pi} = \eta(G:M_1)_{\pi}$ for any M_1, M_2 in $\lambda(G)$.

PROOF. Let the condition of the theorem hold. If $|G|_{\pi} = 1$ then G is a π' -group and hence it is π -solvable. So we assume that $|G|_{\pi} \neq 1$. Let G be simple and $p_1, p_2, \dots, p_n$ be the set of prime divisors of $|G|$, which belong to π . Then as in the proof of Theorem 3.4, we can show that there exist maximal subgroups M_i of G such that $[G:M_i]_{p_i} = 1$ ($i = 1, 2, \dots, n$).

By hypothesis, $|G|_{\pi} = [G:M_1]_{\pi} = [G:M_2]_{\pi} = \dots = [G:M_n]_{\pi}$. As each $p_i \in \pi$, it follows that $|G|_{\pi} = 1$, a contradiction. So G can not be simple. Let N be a minimal normal subgroup of G . If $\lambda(G/N)$ is empty then $\Lambda(G/N)$ is also empty and so by definition, $L(G/N) = G/N$ and consequently by the supersolvability of the group $L(G/N)$, it follows that G/N is π -solvable. If $\lambda(G/N)$ consists of only one element M/N , say, then either $\Lambda(G/N)$ is empty or $\Lambda(G/N) = \{M/N\}$. If $\Lambda(G/N)$ is empty then as above G/N is supersolvable. If $\Lambda(G/N) = \{M/N\}$ then $M/N = L(G/N)$ and consequently M/N is normal in G/N . So by Theorem 1 [9], $\eta(G/N:M/N) = [G/N:M/N] = a$ prime, a contradiction, since $M/N \in \Lambda(G/N)$. We now assume that $|\lambda(G/N)| \geq 2$. It can be shown that G/N satisfies the hypothesis of the theorem. So by induction, G/N is π -solvable. As before, we can assume that N is the unique minimal normal subgroup of G . Also we see that $L(G) \neq G$. If $N \leq L(G)$ then N is solvable and hence it is π -solvable. If $N \not\leq L(G)$ then there exists M_0 in $\Lambda(G)$ such that $N \not\leq M_0$ and so $G = M_0 N$ and $\text{core}_G(M_0) = \langle 1 \rangle$. Let M be any maximal subgroup of G with core 1. Then $N \not\leq M$ and so $G = MN$. Consequently $\eta(G:M) = |N| = \eta(G:M_0)$, whence it follows that M belongs to $\lambda(G)$. By hypothesis $[G:M]_{\pi} = |N|_{\pi}$. If $|N|_{\pi} = 1$ then N is π -solvable. If $|N|_{\pi} \neq 1$ then using (2.4), we have N is solvable and hence it is π -solvable. Thus G/N and N are both π -solvable and consequently G is π -solvable.

The converse follows directly from Theorem 5 [9].

THEOREM 3.7. Let G be a p -solvable group and $|\Sigma_p(G)| \geq 2$. Then G is π -solvable if and only if $\eta(G:M_1)_{\pi} = \eta(G:M_2)_{\pi}$ implies $[G:M_1]_{\pi} = [G:M_2]_{\pi} = \eta(G:M_1)_{\pi}$ for any M_1, M_2 in $\Sigma_p(G)$.

THEOREM 3.8. Let G be a p -solvable group and $|\Sigma_p(G)| \geq 2$. Then G is π -solvable if and only if the following hold.

- (i) G has a π -solvable maximal subgroup M with $\eta(G:M)_{\pi} = [G:M]_{\pi}$.
- (ii) $\eta(G:M_1)_{\pi} = \eta(G:M_2)_{\pi}$ implies $[G:M_1]_{\pi} = [G:M_2]_{\pi}$ for any M_1, M_2 in $\Sigma_p(G)$.

THEOREM 3.9. Let G be a group with $|\lambda(G)| \geq 2$. Then G is π -solvable if and only if the following hold.

- (i) G has a π -solvable maximal subgroup M with $\eta(G:M)_{\pi} = [G:M]_{\pi}$.
- (ii) $\eta(G:M_1)_{\pi} = \eta(G:M_2)_{\pi}$ implies $[G:M_1]_{\pi} = [G:M_2]_{\pi}$ for any M_1, M_2 in $\lambda(G)$.

PROPOSITION 3.10. Let G be a p -solvable group and $|\Sigma_p(G)| \geq 2$. Then G is π -solvable if $\eta(G:M_1)_{\pi} = \eta(G:M_2)_{\pi} = 1$ for all M_1, M_2 in $\Sigma_p(G)$ with equal normal index.

PROPOSITION 3.11. Let G be a group with $\Lambda(G) \neq \emptyset$. Then G is π -solvable if $\eta(G:M)_{\pi} = 1$ for each M in $\Lambda(G)$.

PROPOSITION 3.12. Let G be a p -solvable group or p be the largest prime dividing $|G|$ and $\Sigma_p(G) \neq \emptyset$. Then G is π -solvable if $\eta(G:M)_{\pi} = 1$ for each M in $\Sigma_p(G)$.

PROPOSITION 3.13. Let G be a group with $|\lambda(G)| \geq 2$. Then G is π -solvable if $\eta(G:M_1)_{\pi} = \eta(G:M_2)_{\pi} = 1$ for all M_1, M_2 belonging to $\lambda(G)$ with equal normal index.

PROPOSITION 3.14. If a group G has a π -solvable maximal subgroup M with $\eta(G:M)_{\pi} = 1$ then G is π -solvable.

PROOF. Let G satisfy the hypothesis of the proposition. Then G is not simple. For, otherwise, $|G|_{\pi} = \eta(G:M)_{\pi} = 1$ and so G is π -solvable. Let N be a minimal normal subgroup of G . If $N \leq M$ then N is π -solvable and also, by induction, G/N is π -solvable and hence G is π -solvable. If $N \not\leq M$ then $G = MN$ and since $G/N \cong M/M \cap N$, G/N is π -solvable. Also by hypothesis $|N|_{\pi} = \eta(G:M)_{\pi} = 1$ and so N is π -solvable. Hence G is π -solvable.

4. SOME RESULTS ON SUPERSOLVABLE GROUPS.

THEOREM 4.1. Let G be a p -solvable group and suppose that for each c -maximal

subgroup M of G , $[G:M]_p = 1$ or p . Then G is supersolvable if and only if $\eta(G:M)$ is square-free for each M in $\Sigma_p(G)$.

PROOF. Let G satisfy the hypothesis of the theorem. We claim that $\Sigma_p(G)$ is empty. If possible, let there exist M in $\Sigma_p(G)$. Then G is not simple. For otherwise, $|G| = \eta(G:M)$ is square-free and so G is supersolvable. Let $\eta(G:M) = |H/K|$, where H/K is a chief factor of G and H is minimal in the set of normal supplements of M in G . By hypothesis $|H/K|$ is square-free and hence H/K is supersolvable. Thus H/K is a solvable minimal normal subgroup of G/K . So H/K is an elementary abelian q -group for some prime q . Consequently $\eta(G:M) = |H/K| = q$, a prime, which is a contradiction. So $\Sigma_p(G)$ is empty. By definition $G = S_p(G)$ and hence G is solvable we shall now show that $\Lambda(G)$ is empty. If possible, let there exist M in $\Lambda(G)$. Then since $\eta(G:M) = [G:M]$, [2, Corollary of Theorem 1], it follows that $\eta(G:M)$ is composite and hence p divides $[G:M]$. Now the solvability of G implies that $[G:M]$ is the power of the prime p . By hypothesis, $[G:M] = [G:M]_p = p$, a prime, which is a contradiction. Hence $\Lambda(G)$ is empty and consequently $G = L(G)$. Hence G is supersolvable.

Conversely if G is supersolvable then $\eta(G:M) = [G:M] = p$ for each maximal subgroup M of G and hence the assertion immediately follows.

PROPOSITION 4.2. Let p, q be two distinct primes. Suppose that G is either p -solvable or q -solvable. Then G is supersolvable if and only if $\eta(G:M)$ is square-free for every M in $\Sigma_p(G)$ or $\Sigma_q(G)$.

PROPOSITION 4.3. If G contains a supersolvable maximal subgroup M such that $\text{core}_G(M) = \langle 1 \rangle$ and $\eta(G:M)$ is square-free then G is supersolvable.

PROOF. Let G be simple. By hypothesis, $|G| = \eta(G:M)$ is square-free. So G is supersolvable. We now assume that G is not simple. Let N be a minimal normal subgroup of G . Since $\text{core}_G(M) = \langle 1 \rangle$, it follows that $N \not\leq M$ and so $G = MN$. By hypothesis $|N| = \eta(G:M)$ is square-free and so N is supersolvable. Since $G/N \cong M/M \cap N$, it follows that G/N is supersolvable. Thus G/N and N are both solvable. Hence G is solvable. Now since N is a minimal normal subgroup of the solvable group, it follows that N is an elementary abelian p -group for some prime p . Hence $|N| = p$ and consequently N is cyclic. Therefore G is supersolvable.

PROPOSITION 4.4. If G contains a supersolvable maximal subgroup M such that $\eta(G:M)$ is square-free and the Fitting subgroup, $F(G)$, is not contained in M then G is supersolvable.

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