RESEARCH NOTES

GENERAL BOUNDEDNESS THEOREMS TO SOME SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION WITH INTEGRABLE FORCING TERM

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ABSTRACT. In this note we present a boundedness theorem to the equation $x'' + c(t, x, x') + a(t)b(x) = e(t)$ where $e(t)$ is a continuous absolutely integrable function over the nonnegative real line. We then extend the result to the equation $x'' + c(t, x, x') + a(t, x) = e(t)$. The first theorem provides the motivation for the second theorem. Also, an example illustrating the theory is then given.

KEY WORDS AND PHRASES. Integrable forcing term, bounded, nonlinear differential equation.

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1. INTRODUCTION.

In this article we shall discuss using standard methods the boundedness properties of a second order nonlinear differential equation with integrable forcing term, i.e. the equation,

$$x'' + c(t, x, x') + a(t)b(x) = e(t) \quad (1.1)$$

Our purpose here is to simplify some of the previous proofs to this well-known equation as well as extending some of the previous results. For example, we are replacing the condition $c(t, x, y)y > 0$ for $y \neq 0$ with $c(t, x, y)y \geq 0$ and letting $a(t)$ be non-increasing (see [1] and [2] for details, especially [2] for its excellent bibliography of previous work). Also, as in [2] we shall not need to make use of any Liapunov function. Finally, the result will be of such a nature that it covers the case when no damping factor appears, i.e. it covers the equation,

$$x'' + a(t)b(x) = e(t) \quad (1.2)$$

Later we shall briefly mention how this result carries over to the more general nonlinear equation,

$$x'' + c(t, x, x') + a(t, x) = e(t) \quad (1.3)$$

However, this case requires a more delicate discussion. We now state and prove the boundedness theorem. Without loss of generality, we shall assume $t \geq 0$.

2. MAIN RESULTS.

THEOREM I. Given the differential equation in (1.1). Suppose $c(t, x, y)$ is continuous on $[0, \infty) \times R \times R$, $c(t, x, y)y \geq 0$ and $e(\cdot)$ is continuous on $[0, \infty)$ with $\int_0^\infty |e(t)|dt < \infty$. Furthermore, if $a(\cdot) \geq a_0 > 0$ for some a_0 and continuous on $[0, \infty)$, $a'(\cdot) \leq 0$, $b(\cdot)$ continuous on R , and $B(x) = \int_0^x b(u)du$ approaches ∞ as $|x| \rightarrow \infty$ then all solutions as well as their derivatives are bounded as $t \rightarrow \infty$.

PROOF. By standard existence theory, there is a solution to (1) which exists on $[0, T)$ for some $T > 0$ for any initial conditions $x(0)$ and $x'(0)$. Multiply equation (1) by x' and perform an integration by parts on the last term from 0 to $t < T$ in order to obtain,

$$\begin{aligned}
 x'(t)^2/2 + \int_0^t c(s, x(s), x'(s))x'(s)ds + a(t)B(x(t)) - \int_0^t a'(s)B(x(s))ds \\
 = x'(0)^2/2 + \int_0^t e(s)x'(s)ds \leq x'(0)^2/2 + \int_0^t |e(s)x'(s)|ds
 \end{aligned} \quad (2.1)$$

Now if $x(t)$ becomes unbounded then we must have that all terms on the LHS of (2.1) become positive from our hypotheses. By the mean value theorem, equation (2.1) may be rewritten as,

$$\begin{aligned}
 x'(t)^2/2 + \int_0^t c(s, x(s), x'(s))ds + a(t)B(x(t)) - \int_0^t a'(s)B(x(s))ds \\
 \leq x'(0)^2/2 + |x'(\bar{t})|K \left(K = \int_0^\infty |e(t)|dt, 0 < \bar{t} < t \right)
 \end{aligned} \quad (2.2)$$

Now from (2.2) we see that if $|x|$ approaches ∞ then so must $|x'(t)|$. Otherwise, the LHS of (2.2) becomes unbounded while the RHS stays bounded which is impossible. Also, as $|x'(t)|$ approaches ∞ so must $|x'(\bar{t})|$. Now on any compact subinterval choose t where $x'(t)$ is a maximum. Integrate equation (1.1) as before from 0 to t and divide by $x'(t)$ (assume $x(t) > 0$, a similar argument works for $x'(t) < 0$ only the inequality is reversed) in order to obtain,

$$\begin{aligned}
 x'(t)/2 + 1/x'(t) \left(\int_0^t c(s, x(s), x'(s))ds + a(t)B(x(t)) - \int_0^t a'(s)B(x(s))ds \right) \\
 \leq (x'(0)^2/2 + |x'(\bar{t})|K)/x'(t)
 \end{aligned} \quad (2.3)$$

Now if $x'(t)$ approaches ∞ then the LHS of (2.3) becomes unbounded while the RHS of (2.3) stays bounded which is a contradiction. Thus, $|x|$ and $|x'|$ must stay bounded on $[0, T)$. A standard argument ([3, pp. 17-18]) now permits the solution to be extended on all of $[0, \infty)$.

As for equation (1.3) we may multiply it by x' and integrate as before obtaining the following,

$$\begin{aligned}
 x'(t)^2/2 + \int_0^t c(s, x(s), x'(s))x'(s)ds + \int_{x(0)}^{x(t)} a(t, u)du - \int_0^t \int_{x(0)}^{x(s)} \frac{\partial a(s, u(s))}{\partial s} dud s \\
 = x'(0)^2/2 + \int_0^t e(s)x'(s)ds.
 \end{aligned} \quad (2.4)$$

We see here that as long as $\int_0^{\pm\infty} a(t, u)du = \infty$ uniformly in t and $x \frac{\partial}{\partial t} a(t, x) \leq 0$ then we may use the same argument as in our first theorem. We now state this final result.

THEOREM II. Given equation (1.3). Suppose $c(t, x, y)$ is continuous on $[0, \infty) \times R \times R$, $c(t, x, y)y > 0$, $a(t, x)$ continuous on $[0, \infty) \times R$ with $x \frac{\partial}{\partial t} a(t, x) \leq 0$. Furthermore, if $\int_0^{\pm\infty} a(t, u)du = \infty$ uniformly in t and $e(\cdot)$ is continuous on $[0, \infty)$ with $\int_0^\infty |e(t)|dt < \infty$, then all solutions to equation (1.3) as well as their derivatives are bounded as $t \rightarrow \infty$.

EXAMPLE. Consider the nonlinear differential equation,

$$x'' + cx^{2m-1}x' + bx^{2n-1} = \exp(-t) \quad (2.5)$$

where $t > 0$, c, b are positive and m, n are positive integers. By Theorem I we see that all solution to equation (2.5) are bounded.

REFERENCES

- [1] ANTOSIEWICZ, H.A., On nonlinear differential equations of second order with integrable forcing term, *J. Lond. Math. Soc.*, **30** (1955), 64-67.
- [2] ATHANASOV, Z.S., Boundedness criteria for solutions of certain second order nonlinear differential equations, *J. Math. Anal. Appl.*, **123** (1987), 461-479.
- [3] HALE, J., *Ordinary Differential Equations*, Interscience, New York, 1969.