ABOUT THE EXISTENCE AND UNIQUENESS THEOREM FOR HYPERBOLIC EQUATION

M. E. KHALIFA
Department of Mathematics
Faculty of Science
Benha University
Benha, Egypt

(Received August 29, 1989 and in revised form April 17, 1992)

ABSTRACT. In this paper we prove the existence and uniqueness theorem for almost everywhere solution of the hyperbolic equation using the method of successive approximations [1].

KEY WORDS AND PHRASES. Hyperbolic equation, existence and uniqueness.

1991 AMS SUBJECT CLASSIFICATION CODE. 35H05.

1. INTRODUCTION.

Mixed problems for partial differential equations have been investigated by a number of authors [2], [3], [4], [5]. In this case we investigate the almost everywhere solution for the hyperbolic equation that have been studied in [6]. Namely, the solution for the hyperbolic equation in the space $B^{\frac{3}{2},1}_{2,2,T}$ with a nonlinear operator at the right hand side.

2. STATEMENT OF THE PROBLEM.

Consider the following system

$$u_{tt}(t,x) - Lu(t,x) = F(u(t,x)) \quad \text{in } QT$$

subject to the initial conditions

$$u(0,x) = \phi(x) \quad u_t(0,x) = \psi(x) \quad x \in \Omega,$$ 

and the boundary condition

$$u(t,x) |_{\Gamma} = 0 \quad t \in [0,T]$$

where $QT = [0,T] \times \Omega, 0 < T < \infty, \Omega$ is a bounded domain in $R^n$ and $G$ is the boundary of $O$;

$$L(u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) - a(x)u,$$

and moreover the functions $a_{ij}(x)$ have continuous $\Omega$ and $\frac{\partial a_{ij}(x)}{\partial x_k}, a(x)$ are measurable and bounded in $\Omega$ and satisfy the following conditions in $\Omega$ :

$$a_{ij}(x) = a_{ji}(x), \quad a(x) \geq 0, \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \geq a \sum_{i=1}^{n} \xi_i^2,$$

$\xi_i$ are any real number; $\phi(x), \psi(x)$ are given functions in $\Omega$; $F$ is a nonlinear operator.

3. PRELIMINARIES.
DEFINITION 1. The almost everywhere solution for the problem (2.1)-(2.3) is the function \( u(x,t) \), element of \( W^2_0(Q_T) \), belongs to \( D_0^1(Q_T) \) and satisfies (2.1) almost everywhere in \( Q_T \) and \( t \to +0 \) satisfies the following

\[
\int_{Q_T} [u(t,x) - \phi(x)]^2 dx = 0, \quad \int_{Q_T} \left( \frac{\partial u(t,x)}{\partial t} - \psi(x) \right)^2 dx = 0
\]  

(2.6)

DEFINITION 2. We define the space \( B_{\alpha_0,\ldots,\alpha_\ell}^{0,\ldots,\ell} \) of all functions \( u(t,x) = \sum_{s=1}^\infty \sum_{\ell=1}^\infty \beta_s \psi_{\ell}(x) \) in \( Q_T = [0,T] \times \Omega \), where \( \psi_{\ell}(x) \) are eigenfunctions for the operator \( L \) with the boundary condition (2.3) corresponding to the eigenvalues \( \lambda_s \)

\[
0 < \lambda_s \rightarrow \text{as } s \to \infty \]  

(2.7)

and has the norm

\[
\| u \|_{B_{\alpha_0,\ldots,\alpha_\ell}^{0,\ldots,\ell}} = \sum_{i=1}^l \left\{ \sum_{s=1}^\infty \left[ \frac{\ell_s}{0 \leq \ell \leq T} \max_{0 \leq t \leq T} \left| u_s(t) \right| \right] \right\}^{1/\beta_i} < +\infty
\]  

(2.8)

where \( \alpha_i \geq 0, 1 \leq \beta_i \leq 2, (i = 0, \ldots, \ell) \).

DEFINITION 3. The function \( u_\ell(t) \) is called the \( \ell \)-component of the function

\[
u_{\ell}(t,x) = \sum_{s=1}^\infty \sum_{\ell=1}^\infty \beta_s \psi_{\ell}(x) \]

and \( \mu_s (s = 1, 2, \ldots) \) is the set of all \( \ell \)-components of elements of \( \mu \) where \( \mu \subset B_{\alpha_0,\ldots,\alpha_\ell}^{0,\ldots,\ell} \).

THEOREM 2.1. The necessary and sufficient conditions for \( \mu \) to be compact in \( B_{\alpha_0,\ldots,\alpha_\ell}^{0,\ldots,\ell} \) are

(a) for every \( s (s = 1, 2, \ldots) \) the set \( \mu \) is compact in \( C[0,T] \); and

(b) for any given \( \epsilon > 0 \) there exists a natural number \( n_\epsilon \) so that for all \( u(t,x) = \sum_{s=1}^\infty \sum_{\ell=1}^\infty \beta_s \psi_{\ell}(x) \in \mu, \)

\[
\max_{0 \leq t \leq T} \left| u_s(t) \right| < \epsilon.
\]

This theorem can be proved analogously as in ([9] page 277-278).

LEMMA 1. For any almost everywhere solution \( u(t,x) \) of (2.1) - (2.3) functions \( u_{\ell}(t) = \int_{\Omega} u(t,x) \ell_s(x) dx\) satisfy the following system ([7], [8])

\[
u_{\ell}(t) = \psi_{\ell} \cos \lambda_s t + \frac{\psi_{\ell}}{\lambda_s} \sin \lambda_s t + \lambda_s \int_{0}^{t} \int_{\Omega} F(u(r,x)) \ell_s(x) \sin \lambda_s (t-r) dz dr, (s = 1, 2, \ldots)
\]

(2.9)
EXISTENCE AND UNIQUENESS THEOREM FOR HYPERBOLIC EQUATION

143

where

$$\phi_s = \int_{\Omega} \psi(x)\varphi_s(x)dx, \quad \varphi_s = \int_{\Omega} \psi(x)\varphi_s(x)dx.$$ 

3. ASSUMPTION AND RESULTS.

THEOREM 3.1. Let

1. $a_{ij}(x)$ are continuously differentiable on $\Omega$ and $a(x)$ continuous on $\Omega$;
2. The eigenfunctions $\varphi_s$ are twice continuously differentiable on $\Omega$;
3. $\phi(x) W^2_{2,2}(\Omega) \cap D^2(\Omega)$, $\psi(x) \in D^2(\Omega)$;
4. $F: B^1_{2,2,2} \cup (W^2_2(Q_T) \cap B^1_{2,2,2}) \rightarrow W^1_{1,2,2}(Q_T)$ and satisfies

$$\| F(u(t,x)) \|_{W^1_{1,2,2}(Q_T)} \leq c(t) + d(t) \| u \|_{B^1_{2,2,2}} \leq B^1_{2,2,2,2},$$

for all $u \in B^1_{2,2,2}$; where $c(t), d(t) \in L^2(0, T)$.
5. For any $u, v \in \mathfrak{X}_o$ (where $\mathfrak{X}_o$ is the sphere $\| u \|_{B^1_{2,2,2}} \leq C_o$)

$$\| F(u(t,x)) - F(v(t,x)) \|_{W^1_{1,2,2}(Q_T)} \leq g(t) \| u - v \|_{B^1_{2,2,2,2}}, \quad (3.2)$$

where

$$C_o = \left\{ \left[ 2 \left[ \| W(t,x) \|_{W^1_{2,2,2}}^2 + 16T a_{ij}^2 \| c(t) \|_{L^2(0, T)}^2 \right] \exp \left[ 16T a_{ij}^2 \| d(t) \|_{L^2(0, T)}^2 \right] \right\}^{\frac{1}{2}} \leq \left[ \right.$$ \left. \left. \right.$$

and

$$a_{ij}^2 = \max \left\{ \frac{n \cdot \max \left\{ \| a_{ij}(x) \|_{L^2(0, T)} \right\}}{C(\Omega)} \right\} \cdot \| a(x) \|_{L^2(0, T)}, \quad C(\Omega)$$ \right.$$

6. For any $u \in B^1_{2,2,2} \cup (W^2_2(Q_T) \cap B^1_{2,2,2})$ and $t \in [0, T]$, $F(u(t,x)) \in D^2(\Omega)$.

Then the problem (2.1) - (2.3) has a unique solution,

PROOF. Let

$$W(x, t) = \sum_{s=1}^\infty \left( \phi_s \cos \lambda_s t + \frac{\psi_s}{\lambda_s} \sin \lambda_s t \right) \varphi_s(x), \quad (3.4)$$

and

$$PF(u) = \sum_{s=1}^\infty \int_0^t \left\{ F(u(\tau, x)) \cdot \varphi_s(x) \sin \lambda_s(t - \tau) \right\} d\tau \cdot \varphi_s(x). \quad (3.5)$$

From (3.4) and (3.5) let us assume that

$$Q(u) = W + PF(u) \quad (3.6)$$

Then it is easy to see that the operator $Q$ acts in $B^1_{2,2,2}$, and satisfies Lipschitz condition

$$\| Q(u) - Q(v) \|_{B^1_{2,2,2}} \leq 24\overline{T} a_{ij} \| g(t) \|_{L^2(0, T)} \| u - v \|_{B^1_{2,2,2}} \leq 24\overline{T} a_{ij}, \quad (3.7)$$

in the sphere $\mathfrak{X}_o$.

Consider the sequence $u_k(t, x) = Q(u_{k-1}(t, x))$ in $B^1_{2,2,2}$ where $u_0(t, x) = 0$. Using (3.1) and
the mathematical induction we get for any $k(k = 1, 2, 3,...)$ and $t \in [0, T]$:

\[
\| u_k \|_{B_{2,1}^{2,1}}^{2,1} \leq 2 \| W \|_{B_{2,1}^{2,1}}^{2,1} + 8T a_0^2 \int_0^t \| F(u_{k-1}(\tau, x)) \|_{L_2(\Omega)}^2 d\tau
\]

\[
\leq 2 \| W \|_{B_{2,1}^{2,1}}^{2,1} + 16T a_0^2 \left\{ \int_0^t c^2(\tau) d\tau + \int_0^t d^2(\tau) \| u_{k-1} \|_{B_{2,1}^{2,1}}^{2,1} d\tau \right\}
\]

\[
= A^2 + \left\{ \int_0^t \mathcal{B}^2(\tau) d\tau \right\} k^2 - 1 + \left\{ \int_0^t \mathcal{B}^2(\tau) d\tau \right\}
\]

\[
\leq A^2 + \left\{ \int_0^t \mathcal{B}^2(\tau) d\tau \right\} + \left\{ \int_0^t \mathcal{B}^2(\tau) d\tau \right\}
\]

where

\[
A^2 = 2 \| W \|_{B_{2,1}^{2,1}}^{2,1} + 16T a_0^2 \| c(t) \|_{L_2(0, T)}^2,
\]

and

\[
\mathcal{B}^2(t) = 16T a_0^2 d^2(t)
\]

From (3.8) for any $k(k = 1, 2,...)$, we get

\[
\| u_k \|_{B_{2,1}^{2,1}}^{2,1} \leq A^2 \exp \left\{ \int_0^t \mathcal{B}^2(\tau) d\tau \right\} = C_0^2
\]

i.e., all $u_k(t, x)$ are contained in the sphere $\mathcal{K}_0$. Further, using (3.2) and (3.3) we get for any $t \in [0, T]$ and $k(k = 1, 2, 3,...)$

\[
\| u_{k+1} - u_k \|_{B_{2,1}^{2,1}}^{2,1} \leq 4T a_0^2 \| F(u_k(\tau, x)) - F(u_{k-1}(\tau, x)) \|_{L_2(\Omega)}^2 d\tau
\]

\[
\leq 4T a_0^2 \int_0^t g^2(\tau) \| u_k - u_{k-1} \|_{B_{2,1}^{2,1}}^{2,1} d\tau
\]

\[
\leq \| u_1 - u_0 \|_{B_{2,1}^{2,1}}^{2,1} \left\{ \int_0^t g^2(\tau) d\tau \right\}^{k+1} / k!
\]

\[
= \| u_1 \|_{B_{2,1}^{2,1}}^{2,1} \left\{ \int_0^t g^2(\tau) d\tau \right\}^{k+1} / k!
\]

\[
\leq C_0^2 \left\{ \int_0^t g^2(\tau) d\tau \right\}^{k+1} / k!
\]

Therefore,

\[
\| u_{k+1} - u_k \|_{B_{2,1}^{2,1}}^{2,1} \leq C_0^2 \left\{ \int_0^t g(\tau) \|_{L_2(0, T)}^2 \right\}^{k+1} / k!
\]

\[
(k = 1, 2,...)
\]
Then \( \{ u_k(t, z) \} \) is a fundamental sequence in \( B^2,2,T \). Since \( B^2,2,T \) complete, then

\[
\lim_{k \to \infty} u_k(t, z) = u(t, z) \quad \text{as} \quad k \to \infty
\]  
(3.13)

Since \( Q \) is continuous in \( X_0 \), then from the relation \( u_k(t, z) = Q(u_{k-1}(t, z)) \) we have

\[
u(t, z) = Q(u(t, z))
\]

Therefore, as in (3.11), (3.12) the speed of convergence is governed by the following inequality

\[
\| u_k - u \|_{B^2,2,T} \leq \left\{ \frac{4T \alpha_0^2 \| g(t) \|_2 L_2(0,T)}{k!} \right\}^k \leq C_0 \left\{ \frac{4T \alpha_0^2 \| g(t) \|_2 L_2(0,T)}{k!} \right\}^k, \quad (k = 1, 2, ...).
\]  
(3.14)

Now to prove the uniqueness let us assume the \( u(t, z) = \sum_{s=1}^{\infty} u_s(t) \epsilon_2(z) \) solution to (2.1) - (2.3) then \( F(u(t, z)) \in L_2(Q_T) \). By Lemma (1) \( u_s(t) \) satisfy (2.9); from (2.9) we get

\[
\| u(t, z) \|_{B^2,2,T} \leq \| W(t, z) \|_{B^2,2,T} + 2 \sqrt{T} \| F(u(t, z)) \|_{L_2(Q_T)} < +\infty \]  
(3.15)

Therefore \( u \in B^2,2,T \). Since \( u(t, z) \in W^2_2(Q_T) \cap B^2,2,T \) then by (3.1) \( F(u(t, z)) \in W^2_2,2,2(T) \), but by condition 6 Theorem 2 for all \( t \in [0,T], F(u(t, z)) \in \dot{D}(\Omega) \). Thus using (2.9) with some manipulation

\[
\| u(t, z) \|_{B^2,2,T} \leq \| W(t, z) \|_{B^2,2,T} + 2 \sqrt{T} a_0 \| F(u(t, z)) \|_{W^2_2,2,2(T)} < +\infty \]  
(3.16)

Therefore, \( u \in B^2,2,T \). Then, using (3.1), (3.8), (3.10), we get \( \| u(t, z) \|_{B^2,2,T} \leq C_0 \). Thus, all almost everywhere solutions (2.1)-(2.3) belong to the sphere \( K_0 \) and they are fixed points in \( B^2,2,T \) for operator \( Q \). Let \( u,v \) be two solutions to (2.1)-(2.3), then by (3.2) we get

\[
\| u - v \|_{B^2,2,T} \leq 4 \alpha_0 \int_0^t \| F(u(\tau, z)) - F(v(\tau, z)) \|_{W^2_2(\Omega)} d\tau \leq 4 \alpha_0^{\frac{t}{2}} \| u - v \|_{B^2,2,T} \]  
(3.17)

Therefore, using Belmann’s inequality [10] we have

\[
\| u - v \|_{2,2,T} = 0 \quad \text{in} \quad [0,T]. \quad \text{Therefore,} \quad u = v.
\]

REFERENCES


Submit your manuscripts at http://www.hindawi.com