Internat. J. Math. & Math. Sci.
VOL. 18 NO. 1 (1995) 141-146

ABOUT THE EXISTENCE AND UNIQUENESS THEOREM FOR HYPERBOLIC EQUATION

M. E. KHALIFA

Department of Mathematics
Faculty of Science
Benha University
Benha, Egypt

(Received August 29, 1989 and in revised form April 17, 1992)

ABSTRACT. In this paper we prove the existence and uniqueness theorem for almost everywhere solution of the hyperbolic equation using the method of successive approximations [1].

KEY WORDS AND PHRASES. Hyperbolic equation, existence and uniqueness.
1991 AMS SUBJECT CLASSIFICATION CODE. 35H05.

1. INTRODUCTION.

Mixed problems for partial differential equations have been investigated by a number of authors [2], [3], [4], [5]. In this case we investigate the almost everywhere solution for the hyperbolic equation that have been studied in [6]. Namely, the solution for the hyperbolic equation in the space $B^{2,1}_{2,2,T}$ with a nonlinear operator at the right hand side.

2. STATEMENT OF THE PROBLEM.

Consider the following system

$$u_{tt}(t,x) - Lu(t,x) = F(u(t,x)) \qquad \text{in } Q_T \tag{2.1}$$

subject to the initial conditions

$$u(0,x) = \phi(x) \qquad u_t(0,x) = \psi(x) \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$u(t,x) \,|\, \Gamma = 0 \qquad t \in [0,T] \tag{2.3}$$

where $Q_T = [0,T] \times \Omega, 0 < T < \infty, \Omega$ is a bounded domain in R^n and G is the boundary of O;

$$L(u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - a(x)u, \tag{2.4}$$

and moreover the functions $a_{ij}(x)$ have continuous $\overline{\Omega}$ and $\dfrac{\partial a_{ij}(x)}{\partial x_k}$, $a(x)$ are measurable and bounded in Ω and satisfy the following conditions in Ω :

$$a_{ij}(x) = a_{ji}(x), \quad a(x) \geq 0, \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a\sum_{i=1}^{n}\xi_i^2 , \tag{2.5}$$

ξ_i are any real number; $\psi(\xi)$, $\varphi(x)$ are given functions in Ω; F is a nonlinear operator.

3. PRELIMINARIES.

DEFINITION 1. The almost everywhere solution for the problem (2.1)-(2.3) is the function $u(x,t)$, element of $W_2^2(Q_T)$, belongs to $D_1^0(Q_T)$ and satisfies (2.1) almost everywhere in Q_T and $t \to +0$ satisfies the following

$$\int_\Omega [u(t,x) - \phi(x)]^2 dx = 0 \ , \qquad \int_\Omega [\frac{\partial u(t,x)}{\partial t} - \psi(x)]^2 dx = 0 \tag{2.6}$$

DEFINITION 2. We define the space $B^{\alpha_0,\dots,\alpha_\ell}_{\beta_0,\dots,\beta_\ell,T}$ of all functions $u(t,x) = \sum_{s=1}^{\infty} u_s(t)\vartheta_s(x)$ in $Q_T = [0,T]\times\Omega$, where $v_s(x)$ are eigenfunctions for the operator L with the boundary condition (2.3) corresponding to the eigenvalues λ_s

$$(0 < \lambda_s \to \text{ as } s \to \infty)\ [7]\ , \qquad u_s(t) \text{ are } \ell \geq 0$$

times continuously differentiable in $[0,T]$ and

$$\sum_{i=1}^{l} \left\{ \sum_{s=1}^{\infty} \left[\lambda_s^{\alpha_i} \max_{0 \leq t \leq T} | u_s^{(i)}(t) | \right]^{\beta_i} \right\}^{1/\beta_i} < +\infty \tag{2.7}$$

and has the norm

$$\| u \|_{B^{\alpha_0,\dots,\alpha_\ell}_{\beta_0,\dots,\beta_\ell,T}} = \sum_{i=1}^{\ell} \left\{ \sum_{s=1}^{\infty} \left[\lambda_s^{\alpha_i} \max_{0 \leq t \leq T} | u_s^{(i)}(t) | \right]^{\beta_i} \right\}^{1/\beta_i} \tag{2.8}$$

where $\alpha_i \geq 0, 1 \leq \beta_i \leq 2, (i = 0,\dots,\ell)$.

DEFINITION 3. The function $u_s(t)$ is called the s-component of the function

$$u,(t,x) = \sum_{s=1}^{\infty} u_s(t)\ell_s(x)$$

and $\mu_x(s = 1,2,\dots)$ is the set of all s-components of elements of μ where $\mu \subset B^{\alpha_0,\dots,\alpha_\ell}_{\beta_0,\dots,\beta_\ell,T}$.

THEOREM 2.1. The necessary and sufficient conditions for μ to be compact in $B^{\alpha_0,\dots,\alpha_\ell}_{\beta_0,\dots,\beta_\ell,T}$ are

(a) for every $s(s = 1,2,\dots)$ the set μ_s is compact in $C^\ell[0,T]$; and

(b) for any given $\epsilon > 0$ there exists a natural number n_ϵ so that for all $u(t,x) = \sum_{s=1}^{\infty} u_s\ell_s(x) \in \mu$,

$$\sum_{i=1}^{\ell} \left\{ \sum_{s=n_\epsilon}^{\infty} \left[\lambda_s^{\alpha_i} \max_{0 \leq t \leq T} | u_s^{(i)}(t) | \right]^{\beta_i} \right\}^{1/\beta_i} < \epsilon.$$

This theorem can be proved analogously as in ([9] page 277-278).

LEMMA 1. For any almost everywhere solution $u(t,x)$ of (2.1) - (2.3) functions $u_s(t) = \int_\Omega u(t,x)\ell_s(x)dx$ satisfy the following system ([7] , [8])

$$u_s(t) = \phi_s \cos \lambda_s t + \frac{\psi_s}{\lambda_s} \sin \lambda_s t +$$

$$\frac{1}{\lambda_s} \int_0^t \int_\Omega F(u(\tau,x)).\ell_s(x) \sin \lambda_s(t-\tau) dx d\tau, (s = 1,2,\dots), \tag{2.9}$$

where

$$\phi_s = \int_\Omega \varphi(x)\vartheta_s(x)\,dx\,, \qquad \psi_s = \int_\Omega \psi(x)\vartheta_s(x)\,dx\,.$$

3. ASSUMPTION AND RESULTS.

THEOREM 3.1. Let

1. $a_{ij}(x)$ are continuously differentiable on $\bar{\Omega}$ and $a(x)$ continuous on $\bar{\Omega}$;
2. The eigenfunctions ϑ_s are twice continuously differentiable on $\bar{\Omega}$;
3. $\varphi(x)\epsilon W_2^2(\Omega)\cap D^o(\Omega)$, $\psi(x)\in D^o(\Omega)$;
4. $F: B^{2,1}_{2,2,T}\cup(W_2^2(Q_T)\cap B^{1,0}_{2,2,T})\to W^{1,0}_{x,t,2}(Q_T)$ and satisfies

$$\| F(u(t,x))\|_{W^1_2(\Omega)} \le c(t)+d(t)\,\|u\|_{B^{2,1}_{2,2,t}} \tag{3.1}$$

for all $u\in B^{2,1}_{2,2,T}$; where $c(t),d(t)\in L_2(0,T)$.

5. For any $u,v\in\mathcal{K}_o$ (where $\mathcal{K}_o$ is the sphere $\|u\|_{B^{2,1}_{2,2,T}}\le C_o$)

$$\| F(u,t,x))-F(v(t,x)\|_{W^1_2(\Omega)} \le g(t)\,\|u-v\|_{B^{2,1}_{2,2,t}}\,, \qquad g(t)\in L_2(0,T), \tag{3.2}$$

where

$$C_o=\left\{\left[2\|W(t,x)\|^2_{B^{2,1}_{2,2,T}}+16Ta_o^2\,\|c(t)\|^2_{L_2(0,T)}\right]\exp\left[16\,Ta_o^2\,\|d(t)\|^2_{L_2(0,T)}\right]\right\}^{1/2} \tag{3.3}$$

and

$$a_o^2=\max\left\{n\cdot\max_{ij}\left\{\|a_{ij}(x)\|_{C(\bar{\Omega})}\right\},\ \|a(x)\|_{C(\bar{\Omega})}\right\}$$

6. For any $u\in B^{2,1}_{2,2T}\cup\left(W_2^2(Q_T)\cap B^{1,0}_{2,2,T}\right)$ and $t\in[0,T]\,F(u(t,x))\in D^o(\Omega)$.

Then the problem (2.1) - (2.3) has a unique solution,

PROOF. Let

$$W(x,t)=\sum_{s=1}^{\infty}(\phi_s\cos\lambda_s t+\frac{\psi_s}{\lambda_s}\sin\lambda_s t)\vartheta_s(x)\,, \tag{3.4}$$

and

$$PF(u)=\sum_{s=1}^{\infty}\frac{1}{\lambda_s}\int_0^t\int_\Omega F(u(\tau,x))\cdot\vartheta_s(x)\sin\lambda_s(t-\tau)\,dx\,d\tau\cdot\vartheta_s(x) \tag{3.5}$$

From (3.4) and (3.5) let us assume that

$$Q(u)=W+PF(u) \tag{3.6}$$

Then it is easy to see that the operator Q acts in $B^{2,1}_{2,2,T}$ and satisfies Lipschitz condition

$$\|Q(u)-Q(v)\|_{B^{2,1}_{2,2t}}\le 2\sqrt{T}a_o\,\|g(t)\|_{L_2(0,T)}\,\|u-v\|_{B^{2,1}_{2,2,t}} \tag{3.7}$$

in the sphere $\mathcal{K}_o$.

Consider the sequence $u_k(t,x)=Q(u_{k-1}(t,x))$ in $B^{2,1}_{2,2,T}$ where $u_o(t,x)=0$. Using (3.1) and

the mathematical induction we get for any $k(k=1,2,3,...)$ and $t\in[0,T]$:

$$\| u_k \|^2_{B^{2,1}_{2,2,t}} \le 2\| W \|^2_{B^{2,1}_{2,2,T}} + 8Ta_o^2\int_0^t \| F(u_{k-1}(\tau,x)) \|^2_{W^1_2(\Omega)} d\tau$$

$$\le 2\| W \|^2_{B^{2,1}_{2,2,T}} + 16Ta_o^2\left\{\int_0^T c^2(\tau)d\tau + \int_0^t d^2(\tau)\| u_{k-1} \|^2_{B^{2,1}_{2,2,t}} d\tau\right\} \tag{3.8}$$

$$= A^2 + \int_0^t \mathfrak{B}^2(\tau)\| u_{k-1} \|^2_{B^{2,1}_{2,2,t}} d\tau$$

$$\le \mathcal{A}^2 + \mathcal{A}^2\int_0^t \mathfrak{B}^2(\tau)d\tau + ... + \mathcal{A}^2\frac{\left\{\int_0^t \mathfrak{B}^2(\tau)d\tau\right\}^{k-1}}{(k-1)!},$$

where

$$\mathcal{A}^2 = 2\| W \|^2_{B^{2,1}_{2,2,T}} + 16Ta_o^2\| c(t) \|^2_{L_2(0,T)}, \tag{3.9}$$

and

$$\mathfrak{B}^2(t) = 16Ta_o^2 d^2(t)$$

From (3.8) for any $k(k=1,2,...)$, we get

$$\| u_k \|^2_{B^{2,1}_{2,2,t}} \le \mathcal{A}^2\cdot \exp\left\{\int_0^T \mathfrak{B}^2(\tau)d\tau\right\} = C_o^2 \tag{3.10}$$

i.e., all $u_k(t,x)$ are contained in the sphere $\mathfrak{K}_o$. Further, using (3.2) and (3.3) we get for any $t\in[0,T]$ and $k(k=1,2,3,...)$

$$\| u_{k+1} - u_k \|^2_{B^{2,1}_{2,2,t}} \le 4Ta_o^2\| F(u_k(\tau,x)) - F(u_{k-1}(\tau,x)) \|^2_{L_2(\Omega)} d\tau$$

$$\le 4Ta_o^2\int_0^t g^2(\tau)\| u_k - u_{k-1} \|^2_{B^{2,1}_{2,2,t}}$$

$$\le \| u_1 - u_o \|^2_{B^{2,1}_{2,2,T}} \frac{\left\{4Ta_o^2\int_0^t g^2(\tau)d\tau\right\}^k}{k!}$$

$$= \| u_1 \|^2_{B^{2,1}_{2,2,T}} \frac{\left\{4Ta_o^2\int_0^t g^2(\tau)d\tau\right\}^k}{k!} \le C_o^2\frac{\left\{4Ta_o^2\int_0^t g^2(\tau)d\tau\right\}^k}{k!} \tag{3.11}$$

Therefore,

$$\| u_{k+1} - u_k \|^2_{B^{2,1}_{2,2,T}} \le C_o^2\frac{\left\{4Ta_o^2\| g(t) \|^2_{L_2(0,T)}\right\}^k}{k!}, (k=1,2,...) \tag{3.12}$$

Then $\{u_k(t,x)\}$ is a fundamental sequence in $B^{2,1}_{2,2,T}$. Since $B^{2,1}_{2,2,T}$ complete, then

$$u_k(t,x) \quad \underset{\rightarrow}{B^{2,1}_{2,2,T}} \quad u(t,x) \in \mathfrak{K}_o \qquad \text{as } k \to \infty \tag{3.13}$$

Since Q is continuous in $\mathfrak{K}_o$, then from the relation $u_k(t,x) = Q(u_{k=1}(t,x))$

we have

$$u,(t,x) = Q(u(t,x))$$

Therefore, as in (3.11), (3.12) the speed of convergence is governed by the following inequality

$$\| u_k - u \|^2_{B^{2,1}_{2,2,T}} \leq \| u_o - u \|^2_{B^{1,0}_{2,2,T}} \frac{\left\{4Ta_o^2 \| g(t) \|^2_{L_2(0,T)}\right\}^k}{k!}$$

$$\leq C_o^2 \frac{\left\{4Ta_o^2 \| g(t) \|^2_{L_2(0,T)}\right\}^k}{k!}, (k = 1,2,...). \tag{3.14}$$

Now to prove the uniqueness let us assume the $u(t,x) = \sum_{s=1}^{\infty} u_s(t)\ell_2(x)$ solution to (2.1) - (2.3) then $F(u(t,x)) \in L_2(Q_T)$. By Lemma (1) $u_s(t)$ satisfy (2.9); from (2.9) we get

$$\| u(t,x) \|_{B^{1,0}_{2,2,t}} \leq \| W(t,x) \|_{B^{1,0}_{2,2,T}} + 2\sqrt{T} \| F(u(t,x)) \|_{L_2(Q_T)} < +\infty \tag{3.15}$$

Therefore $u \in B^{1,0}_{2,2,t}$. Since $u(t,x) \in W^2_2(Q_T) \cap B^{1,0}_{2,2,T}$ then by (3.1) $F(u(t,x)) \in W^{1,0}_{x,t,2}(Q_T)$, but by condition 6 Theorem 2 for all $t \in [0,T], F(u(t,x)) \in \overset{o}{D}(\Omega)$. Thus using (2.9) with some manipulation

$$\| u(t,x) \|_{B^{2,1}_{2,2,t}} \leq \| W(t,x) \|_{B^{2,1,}_{2,2,T}} + 2\sqrt{T}\, a_o \| F(u(t,x)) \|_{W^{1,0}_{x,t,2}(Q_T)} < +\infty \tag{3.16}$$

Therefore, $u \in B^{2,1}_{2,2,T}$. Then, using (3.1), (3.8), (3.10), we get $\| u(t,x) \|_{B^{2,1}_{2,2,t}} \leq C_o$. Thus, all almost everywhere solutions (2.1)-(2.3) belong to the sphere K_o and they are fixed points in $B^{2,1}_{2,2,T}$ for operator Q. Le u, v be two solutions to (2.1)-(2.3), then by (3.2) we get

$$\| u - v \|^2_{B^{2,1}_{2,2,t}} \leq 4Ta_o^2 \int_0^t \| F(u(\tau,x)) - F(v(\tau,x)) \|^2_{W^1_2(\Omega)}\, d\tau$$

$$\leq 4Ta_o^2 \int_0^t g^2(\tau) \| u - v \|^2_{B^{2,1}_{2,2,t}}\, d\tau \tag{3.17}$$

Therefore, using Belmann's inequality [10] we have

$$\| u - v \|^2_{2,2,t} = 0 \text{ in } [0,T]. \text{ Therefore, } u = v\,.$$

REFERENCES

1. KRASNOSELSKII, M.A., "Topological methods in the theory of nonlinear integral equations." Pergamon Press, New York, 1964.
2. BURSKII, V.P., "Remarks on the kernel of a differential operator with constant coefficients in a domain.", Mat. Fix. Nelinein. Mekh. No. 2, 36 (1984), 43-45.

3. KAZARYAN, G.G., "Weak solutions of the Dirichlet problem for a quasilinear equation with lower-order terms." Trudy Math. Inst. Steklov. 170, (1984), 105-112.

4. KHALIFA, M.E., "The mixed problem of second order hyperbolic-parabolic system." Dokl. Akad. Nauk Az. USSR, No. 5 (1979), 60-65.

5. KHALIFA, M.E., "The continuously dependent generalized solution for a mixed problem of the hyperbolic-parabolic system." Azerbaizan Gos. Univ. Ucen. Zap. 3 (1979) 65-69.

6. KHALIFA, M.E., "Investigation of existence and uniqueness theorem for hyperbolic equation." Mans. Sci., Bull. Vol. 16(2).

7. LADYZENSKAJA, O.A., "Boundary value problems of mathematical physics." Nauk, Moscow, 1973.

8. LADYZENSKAJA, O.A., "The mixed problem for hyperbolic equations, Gostekhizdat, 1953.

9. SMIRNOV, V.I., "A course of Higer mathematics." Pergamon Press, New York, 1964.

10. BECKENBACH, E.F. and BELLMAN, R., Inequalities, Springer-Verlag, Berlin, 1961.