COMMON FIXED POINTS FOR NONEXPANSIVE AND NONEXPANSIVE TYPE FUZZY MAPPINGS

BYUNG SOO LEE
Department of Mathematics, Kyungsun University
Pusan, 608-736, KOREA

DO SANG KIM
Department of Applied Mathematics
National Fisheries University of Pusan
608-737, KOREA

GUE MYUNG LEE and SUNG JIN CHO
Department of National Sciences, Pusan National University of Technology
Pusan, 608-739, KOREA

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ABSTRACT. In this paper we define $g$-nonexpansive and $g$-nonexpansive type fuzzy mappings and prove common fixed point theorems for sequences of fuzzy mappings satisfying certain conditions on a Banach space. Thus we obtain fixed point theorems for nonexpansive type multi-valued mappings.

KEY WORDS AND PHRASES: Star-shaped set, Opial's condition, weak convergence, Hausdorff metric, nonexpansive fuzzy mapping, nonexpansive type fuzzy mapping, fixed point, common fixed point.


1. INTRODUCTION

Fixed point theorems for fuzzy mappings were obtained by Chang, Heilpern and others [1-5, 7, 9-13, 16]. Especially, Lee and Cho [10] showed that a sequence of fuzzy mappings with the condition (*) satisfies the condition (**), that a sequence with the condition (**) has a common fixed point and consequently that a sequence of fuzzy mappings with the condition (*) has a common fixed point. These results are fuzzy analogues of common fixed theorems for sequences of $g$-contractive and $g$-contractive type multi-valued mappings [8]. In [11] and [13] Lee et al. also obtained a common fixed point theorem for sequences of fuzzy mappings which generalize the results in [1] and [10] respectively.

In this paper we define $g$-nonexpansive and $g$-nonexpansive type fuzzy mappings and show that a sequence of fuzzy mappings with the condition (***)*, which are defined on a nonempty weakly compact star-shaped subset of a Banach space $X$ satisfying Opial's condition, has a common fixed point. As corollaries, firstly we show that similar results are obtained for the conditions (*)$_1$ (**), or (***)$. Secondly we obtain fixed point theorems for nonexpansive type fuzzy [respectively, compact-valued] mappings $F$ [resp., $f$] from $K(cX)$ to $W(K)$ [resp., $2^X$]. Thirdly we show that similar results are obtained for nonexpansive fuzzy [resp., compact-valued] mappings.

2. PRELIMINARIES

We review briefly some definitions and terminologies needed.

A fuzzy set $A$ in a metric space $X$ is a function with domain $X$ and values in $[0,1]$. (In particular, if $A$ is an ordinary (crisp) subset of $X$, its characteristic function $\chi_A$ is a fuzzy set with domain $X$ and values $\{0,1\}$.) Especially $\{x\}$ is a fuzzy set with a membership function equal to a characteristic function of the set $\{x\}$. The $\alpha$-level set of $A$, denoted by $A_\alpha$, is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\} \quad \text{if} \quad \alpha \in (0,1],$$

$$A_0 = \{x : A(x) > 0\}$$

where $\overline{B}$ denotes the closure of the (nonfuzzy) set $B$. 

$W(X)$ denotes the collection of all fuzzy sets $A$ in $X$ such that (i) $A_\alpha$ is compact in $X$ for each $\alpha \in [0,1]$ and (ii) $A_\alpha$ is a nonempty subset of $X$. For $A, B \in W(X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.

Let $A$ and $B$ be two nonempty bounded subsets of a Banach space $X$. The Hausdorff distance between $A$ and $B$ is

$$d_h(A,B) = \max \left[ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right].$$

**Definition 2.1.** Let $A, B \in W(X)$ and $\alpha \in [0,1]$. Then we define

$$D(A,B) = \sup_a d_h(A_\alpha,B_\alpha).$$

We note that $D$ is a metric on $W(X)$ such that $D(\{x\},\{y\}) = \|x - y\|$, where $x, y \in X$.

**Definition 2.2.** Let $X$ be an arbitrary set and $Y$ be any metric space. $F$ is called a fuzzy mapping iff $F$ is a mapping from the set $X$ into $W(Y)$.

A fuzzy mapping $F$ is a fuzzy subset on $X \times Y$ with a membership function $F(x)(y)$. The function value $F(x)(y)$ is the grade of membership of $y$ in $F(x)$. In case $X = Y$, $F(x)$ is a function from $X$ into $[0,1]$. Especially for a multi-valued mapping $f : X \rightarrow 2^X$, $\chi_{f(y)}$ is a function from $X$ to $[0,1]$. Hence a fuzzy mapping $F : X \rightarrow W(X)$ is another extension of a multi-valued mapping $f : X \rightarrow 2^X$.

**Definition 2.3.** Let $g$ be a mapping from a Banach space $(X, \| \cdot \|)$ to itself. A fuzzy mapping $F : X \rightarrow W(X)$ is $g$-contractive [respectively, $g$-nonexpansive] if $D(F(x), F(y)) \leq k \cdot \|g(x) - g(y)\|$ for all $x, y \in X$, for some fixed $k$, $0 \leq k < 1$ [resp., $k = 1$].

**Proposition 2.4** [9]. Let $(X, \| \cdot \|)$ be a Banach space, $F : X \rightarrow W(X)$ a fuzzy mapping and $x \in X$, then there exists $u_x \in X$ such that $\{u_x\} \subset F(x)$.

**Definition 2.5.** Let $g$ be a mapping from a Banach space $(X, \| \cdot \|)$ to itself. We call a fuzzy mapping $F : X \rightarrow W(X)$ $g$-contractive type [respectively, $g$-nonexpansive type] if for all $x \in X, \{u_x\} \subset F(x)$ there exists $\{v_x\} \subset F(y)$ for all $y \in X$ such that $D(\{u_x\}, \{v_y\}) \leq k \cdot \|g(x) - g(y)\|$ for some fixed $k$, $0 \leq k < 1$ [resp., $k = 1$].

**Remark.** When $g$ is an identity, a $g$-contractive [respectively, $g$-contractive type, $g$-nonexpansive, $g$-nonexpansive type] fuzzy mapping $F$ is said to be contractive [resp., contractive-type, nonexpansive, nonexpansive type].

**Lemma 2.6.** Let $A, B \in W(X)$. Then for each $\{x\} \subset A$, there exists $\{y\} \subset B$ such that $D(\{x\}, \{y\}) \leq D(A,B)$.

**Proof.** If $\{x\} \subset A$, then $x \in A_\alpha$. By compactness of $B_\alpha$, we can choose a $y \in B_\alpha$, i.e., $\{y\} \subset B$, such that $\|x - y\| \leq d_h(A_\alpha,B_\alpha)$. By the facts $D(\{x\}, \{y\}) = \|x - y\|$ and $d_h(A_\alpha,B_\alpha) \leq D(A,B)$, we have $D(\{x\}, \{y\}) \leq D(A,B)$.

**Proposition 2.7.** Let $g$ be a mapping from a Banach space $(X, \| \cdot \|)$ to itself. If $F : X \rightarrow W(X)$ is a $g$-nonexpansive [respectively, $g$-contractive] fuzzy mapping, then $F$ is $g$-nonexpansive type [resp., $g$-contractive type].

**Proof.** It can be easily proved by Lemma 2.6.

### 3. COMMON FIXED POINTS FOR FUZZY MAPPINGS

For a mapping $g$ of a Banach space $X$ into itself and a sequence $(F_i)_{i=1}^n$ of fuzzy mappings of $X$ into $W(X)$ we consider the following conditions (*)-****(*)

* there exists a constant $K$ with $0 \leq k < 1$ such that for each pair of fuzzy mappings $F_i, F_j : X \rightarrow W(X)$, $D(F_i(x), F_j(y)) \leq k \cdot \|g(x) - g(y)\|$ for all $x, y \in X$. 

(* )
there exists a constant $k$ with $0 < k < 1$ such that for each pair of fuzzy mappings $F_i, F_j : X \to W(X)$ and for any $x \in X, \{u_i\} \subset F_i(x)$ implies that there is $\{v_j\} \subset F_j(y)$ for all $y \in X$ with $D(\{u_i\}, \{v_j\}) \leq k \|g(x) - g(y)\|_2$.

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for each pair of fuzzy mappings $F_i, F_j : X \to W(X)$, and for any $x \in X, \{u_i\} \subset F_i(x)$ implies that there is $\{v_j\} \subset F_j(y)$ for all $y \in X$ with $D(\{u_i\}, \{v_j\}) \leq \|g(x) - g(y)\|$.

It is easily proved that the condition (*) [respectively, (***)] implies the condition (**) [resp., (***)] by Lemma 2.6, but the following example shows that the converses do not hold in general.

**EXAMPLE 3.1.** Let $g$ be an identity mapping from a Euclidean metric space $([0, \infty), \mathbb{R})$ to itself. Let $(F_i)_{i=1}^\infty$ be a sequence of fuzzy mappings from $[0, \infty)$ into $W([0, \infty))$, where $F_i(x) : [0, \infty) \to [0, 1]$ is defined as follows;

$$
\begin{align*}
F_i(x)(z) &= \begin{cases} 
1, & z = 0, \\
0, & z \neq 0,
\end{cases} \\
&= \begin{cases} 
1, & 0 \leq z \leq x/2, \\
1/2, & x/2 < z \leq ix, \\
0, & z > ix.
\end{cases}
\end{align*}
$$

Then the sequence $(F_i)_{i=1}^\infty$ satisfies the condition (***)

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In this section we show that a sequence of fuzzy mappings with the condition (***)

consequently a sequence of fuzzy mappings with the condition (*) (**) or (***) has a common fixed point.

As corollaries we obtain fixed point theorems for nonexpansive type fuzzy [respectively, compact-valued] mappings $F$ [resp., $f$] from a nonempty weakly compact and star-shaped subset $K$ of a Banach space $X$ which satisfies Opial's condition to $W(X)$ [resp., $2^X$].

The results for the nonexpansive compact-valued mappings are the case of replacing convexity with star-shapedness in Theorem 3.5 due to Husain and Latif [8].

Following Nguyen [14] we define: Let $X, Y$ and $Z$ be any nonempty sets, and $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$ where $\mathcal{F}(X)$ is the collection of all fuzzy sets in $X$. If $f : X \to Y$, then the fuzzy set $f(A)$ is defined via the extension principle by $f(A) : Y \to 2^X$ and $f(A)(y) = \sup_{x \in f^{-1}(y)} A(x)$.

If $f : X \times Y \to Z$, then the fuzzy set $f(A, B)$ is defined via the extension principle by $f(A, B) : Z \to 2^X$ and $f(A, B)(z) = \sup_{(x, y) \in f^{-1}(z)} \min\{A(x), B(y)\}$.

**PROPOSITION (NGUYEN).** Let $f : X \times Y \to Z$ and $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$. Then a necessary and sufficient condition for the equality $[f(A, B)]_\alpha = f(A_\alpha, B_\alpha)$ for all $\alpha \in [0, 1]$ is that for all $z \in Z$, $\sup_{(x, y) \in f^{-1}(z)} \min\{A(x), B(y)\}$ is attained.

A subset $K$ of a Banach space $X$ is said to be star-shaped if there exists a point $v \in K$ such that $tv + (1-t)x \in K$ for all $x \in K$ and $0 < t < 1$. The point $v$ is called the star center of $K$.

**THEOREM 3.2** [10]. Let $g$ be a nonexpansive mapping from a complete metric linear space $(X, d)$ to itself. If $(F_i)_{i=1}^\infty$ is a sequence of fuzzy mappings of $X$ into $W(X)$ satisfying the condition (**), then there exists a point $x \in X$ such that $\{x\} \subset \cap_{i=1}^\infty F_i(x)$.
PROPOSITION 3.3. Let \( K \) be a nonempty bounded star-shaped subset of a Banach space \( X \) and \( g \) a nonexpansive mapping from \( X \) into itself. If \( (F_n)^{\alpha n} \) is a sequence of fuzzy mappings of \( K \) into \( W(X) \) satisfying the condition (**), then there exist a sequence \( (x_n)^{\alpha n} \) in \( K \) and a sequence \( (u_n)^{\alpha n} \) in \( X \) satisfying \( \{u_n\} \subset F_n(x_n) \) for all \( n \in \mathbb{N} \) such that \( \|x_n - u_n\| \to 0 \) as \( n \to \infty \).

**Proof.** Let \( x_0 \) be the star-center of \( K \). Choose a real sequence \( (k_n)^{\alpha n} \) such that \( 0 < k_n < 1 \) and \( k_n \to 0 \) as \( n \to \infty \). Then for each \( x \in K \), \( k_n x_0 + (1-k_n)x \in K \). Define a fuzzy mapping \( F^*_n \) of \( K \) into \( W(X) \) by setting \( F^*_n(x) = k_n \{x_0\} + (1-k_n)F_n(x) \) for all \( x \in K \). Then by Proposition 3.3 in [14] it follows that \( \{F^*_n(x)\} \) is a sequence of fuzzy mappings satisfying the condition (**). If we let \( \{u_n\} \subset F^*_n(x) \) for each \( x \in K \), we get \( u_n = k_n x_0 + (1-k_n)v_n \) for some \( v_n \in K \) such that \( \{v_n\} \subset F_n(x) \). Since \( (F_n)^{\alpha n} \), satisfies the condition (**), there exists a \( \{v_n\} \subset F_n(x) \) for all \( y \in K \) such that \( \|v_n - v_m\| \leq \|g(x) - g(y)\| \leq \|x - y\| \).

Let \( u_n = k_n x_0 + (1-k_n)v_n \), clearly by definition of \( F_n(x) \) we get \( \{u_n\} \subset F_n(x) \) and \( \|u_n - y\| = \|(1-k_n)(v_n - v)\| \leq (1-k_n)\|g(x) - g(y)\| \leq (1-k_n)\|x - y\| \) which proves that \( (F^*_n)^{\alpha n} \) is a sequence of fuzzy mappings satisfying the condition (**). The common fixed point theorem for a sequence of fuzzy mappings due to Lee and Cho [10] i.e., Theorem 3.2 guarantees that for each fixed \( n \in \mathbb{N} \), \( (F^*_n)^{\alpha n} \) has a common fixed point in \( K \), say \( \{x_n\} \subset F_n(x) \) for all \( n \in \mathbb{N} \). From the definition of \( F^*_n(x) \) there exists a \( \{u_n\} \subset F_n(x) \) such that \( x_n = k_n x_0 + (1-k_n)u_n \) for all \( n \in \mathbb{N} \) and each fixed \( n \in \mathbb{N} \). Thus \( \|x_n - u_n\| = \|k_n x_0 + (1-k_n)u_n - u_n\| = k_n\|x_0 - u_n\| \). By the definition of \( W(K) \), \( \{u_n\} \subset F_n(x) \) implies \( u_n \in K \). Thus \( \|u_n - x_0\| \) is bounded. So by the fact that \( k_n \to 0 \) as \( n \to \infty \), we have \( \|x_n - u_n\| \to 0 \) as \( n \to \infty \).

We use the following notion due to Opial [15]. A Banach space \( X \) is said to satisfy Opial’s condition [15] if for each \( x \in X \) and each sequence \( (x_n)^{\alpha n} \) weakly convergent to \( x \),

\[
\lim_{n \to \infty} \|x_n - y\| > \lim_{n \to \infty} \|x_n - x\|
\]

for all \( y \neq x \).

PROPOSITION 3.4. Let \( K \) be a nonempty subset of a Banach space \( X \) which satisfies Opial’s condition and \( F \) a \( g \)-nonexpansive type fuzzy mapping of \( K \) into \( W(K) \). Let \( (x_n)^{\alpha n} \) be a sequence in \( K \) which converges weakly to an element \( x \in K \). If \( (y_n)^{\alpha n} \) is a sequence in \( X \) such that \( \{x_n - y_n\} \subset F(x_n) \) and converges to \( y \in X \), then \( \{x_n - y\} \subset F(x) \).

**Proof.** Since \( F \) is a \( g \)-nonexpansive type fuzzy mapping, there exists a \( \{v_n\} \subset F(x) \) such that \( \|x_n - y_n - v_n\| \leq \|g(x_n) - g(x)\| \leq \|x_n - x\| \). It follows that \( \lim_{n \to \infty} \|x_n - y_n - v_n\| \leq \lim_{n \to \infty} \|x_n - x\| \). Since every weakly convergent sequence is necessarily bounded, limits in the proceeding expression are finite. Since \( (v_n)^{\alpha n} \) is a sequence in a compact subset \( \{F(x)\}_\alpha \) of \( X \) for each \( \alpha \in [0,1] \), there is a subsequence of \( (v_n)^{\alpha n} \), also denoted by \( (v_n)^{\alpha n} \), converging to \( v \in \{F(x)\}_\alpha \) for each \( \alpha \in [0,1] \). Hence \( \{v\} \subset F(x) \), therefore

\[
\lim_{n \to \infty} \|x_n - y_n - v_n\| = \lim_{n \to \infty} \|x_n - y_n - v_n - (y + v)\|
\]

\[
\geq \lim_{n \to \infty} \|x_n - (y + v)\| - \|v_n + v - (y + v)\|
\]

\[
\geq \lim_{n \to \infty} \|x_n - (y + v)\| + \lim_{n \to \infty} \|v_n - y + v\|
\]

\[
= \lim_{n \to \infty} \|x_n - (y + v)\|
\]

Thus we have shown that \( \lim_{n \to \infty} \|x_n - x\| \geq \lim_{n \to \infty} \|x_n - (y + v)\| \).
Since \((x_\alpha)_{\alpha=1}^n\) converges to \(x\) weakly, Opial’s condition implies that \(x = y + v\), so \(x - y = v \in [F(x)]\alpha\) for each \(\alpha \in [0,1]\). Hence \(\{x - y\} \subset F(x)\) and the proposition is proved.

**REMARK.** From the above proof it follows that the weak limit of fixed points of a nonexpansive-type fuzzy mapping \(F\) defined on a nonempty subset \(K\) of a Banach space \(X\) satisfying Opial’s condition, in particular for a Hilbert space is also a fixed point of \(F\).

**THEOREM 3.5.** Let \(K\) be a nonempty weakly compact star-shaped subset of a Banach space \(X\) which satisfies Opial’s condition. If \((F_i)_{i=1}^\infty\) is a sequence of fuzzy mappings of \(K\) into \(W(K)\) satisfying the condition (***)\(\), then \((F_i)_{i=1}^\infty\) has a common fixed point.

**PROOF.** Since \(K\) is weakly compact, it is a bounded subset of \(X\). By the Proposition 3.3 there exist a sequence \((x_n)_{n=1}^\infty\) in \(K\) and a sequence \((u_n)_{n=1}^\infty\) in \(X\) satisfying \(\{u_n\} \subset F_i(x_n)\) for all \(i \in \mathbb{N}\) such that \(||x_n - u_n|| \to 0\) as \(n \to \infty\). Put \(y_n = x_n - u_n\). \(K\) being weakly compact, we can find a weakly convergent subsequence \((x_m)_{m=1}^\infty\) of \((x_n)_{n=1}^\infty\). Let \(x_0\) be the weak limit of the sequence \((x_m)_{m=1}^\infty\). Clearly \(x_0 \in K\) and we have \(y_m = x_m - u_m\), \((u_m) \subset F_i(x_m)\) for all \(i \in \mathbb{N}\). Then it follows that \(y_m \to 0\) and by Proposition 3.4 there exists a fixed point \(x_0 \in X\) such that \(\{x_0\} \subset F_i(x_0)\) for all \(i \in \mathbb{N}\).

**THEOREM 3.6.** Let \(K\) be a nonempty weakly compact star-shaped subset of a Banach space \(X\) which satisfies Opial’s condition. If \((F_i)_{i=1}^\infty\) is a sequence of fuzzy mappings of \(K\) into \(W(K)\) satisfying the condition (*)\(\), (***)\(\) or (****)\(\), then \((F_i)_{i=1}^\infty\) has a common fixed point.

**PROOF.** It is proved by the fact that the condition (****) [respectively, (*)] implies the condition (****) [resp., (***)].

If we put \(F_i = F\) for all \(i \in \mathbb{N}\) in Proposition 3.3, then the sequence of fuzzy mappings \((F_i)_{i=1}^\infty\) is a sequence of \(g\)-nonexpansive type fuzzy mappings. Thus we obtain the following corollary for \(g\)-nonexpansive type fuzzy mappings.

**COROLLARY 3.7.** Let \(K\) be a nonempty weakly compact star-shaped subset of a Banach space \(X\) which satisfies Opial’s condition. Then each \(g\)-nonexpansive type fuzzy mapping \(F : K \to W(K)\) has a fixed point.

**COROLLARY 3.8.** Let \(K\) be a nonempty weakly compact star-shaped subset of a Banach space \(X\) which satisfies Opial’s condition. Then each nonexpansive type, compact-valued mapping \(f : K \to 2^X\) has a fixed point.

**PROOF.** Define \(F : K \to W(K)\) by \(F(x) = \chi_{f(x)}\) then \(F\) is a nonexpansive-type fuzzy mapping. By Corollary 3.7 there exists a point \(x \in X\) such that \(\{x\} \subset F(x) = \chi_{f(x)}\) i.e., \(x \in f(x)\).

**THEOREM 3.9.** Let \(K\) be a nonempty weakly compact convex subset of a Banach space \(X\) which satisfies Opial’s condition. Then each nonexpansive type, compact-valued mapping \(f : K \to 2^X\) has a fixed point.

**COROLLARY 3.10.** Let \(K\) be a nonempty weakly compact star-shaped subset of a Banach space \(X\) which satisfies Opial’s condition. Then each nonexpansive fuzzy mapping \(F : K \to W(K)\) has a fixed point.

**COROLLARY 3.11.** Let \(K\) be a nonempty weakly compact star-shaped subset of a Banach space \(X\) having a weakly continuous duality mapping. Then each nonexpansive-type fuzzy mapping \(F : K \to W(K)\) has a fixed point.

**PROOF.** If a Banach space \(X\) admits a weakly continuous duality mapping, then it satisfies Opial’s condition [6].

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REFERENCES


