MINIMAL CR-SUBMANIFOLDS OF A SIX-DIMENSIONAL SPHERE

M. HASAN SHAHID and S. I. HUSAIN

Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia, New Delhi - 110025
INDIA

Department of Mathematics
Aligarh Muslim University
Aligarh, 202002, INDIA

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ABSTRACT. We establish several formulas for a 3-dimensional CR-submanifold of a six-dimensional sphere and state some results obtained by making use of them.

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1. INTRODUCTION. Among all submanifolds of a Kaehler manifold there are three typical classes: the complex submanifolds, the totally real submanifolds and the CR-submanifolds. The notion of a CR-submanifold of a Kaehler manifold was introduced by Bejancu [1] and it includes the other two classes as special cases. A Riemannian submanifold $M$ of an almost Hermitian manifold $M$ is called a CR-submanifold if there exists a pair of orthogonal complementary distribution $D$ and $D⁺⁻$ on $M$ satisfying $JD = D$ and $JD⁺⁻ \subset \nu$, where $\nu$ is the normal bundle of $M$. If $M$ is a real hypersurface of a Kaehler manifold, then $M$ is obviously a CR-submanifold.

It is known that every Kaehler manifold is nearly Kaehler but the converse is not true in general. The most typical example of nearly Kaehler manifolds is a six-dimensional sphere $S^6$. It is because of this nearly Kaehler, non-Kaehler, structure that $S^6$ has attracted attention.

The object of the present paper is to establish several formulas for a 3-dimensional CR-submanifold of a six-dimensional sphere and state some result obtained by making use of them.

2. PRELIMINARIES.

Let $\bar{M}$ be an almost complex manifold with almost complex structure $J$, and Hermitian metric $g$. $\bar{M}$ is called a nearly Kaehler manifold if

\[
(\nabla_X J)(Y) + (\nabla_Y J)(X) = 0
\]

for $X, Y \in (\bar{M})$, where $\nabla$ is Riemannian connection on $\bar{M}$.

In [5], K. Takamatsu and T. Sato proved the following theorem:

THEOREM. Let $\bar{M} = (\bar{M}, J, g)$ be a non-Kaehler, nearly Kaehler manifold of constant holomorphic sectional curvature. Then $\bar{M}$ is a six-dimensional space of positive constant sectional curvature.

If a nearly Kaehler manifold $\bar{M}$ is constant holomorphic sectional curvature $c$, then by the above result, the curvature tensor $\bar{R}$ of $\bar{M}$ is given by

\[
R(X, Y, Z, W) = c(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)).
\]
Let $M$ be an $m$-dimensional CR-submanifolds of a six-dimensional sphere $\bar{M}$ and let us denote by the same $g$ the Riemannian metric tensor field induced on $M$ from that of $\bar{M}$. Let $P$ and $Q$ be the projection operators corresponding to $D$ and $D^\perp$ respectively.

For a vector field $X$ tangent to $M$, we put

$$JX = PX + QX$$

where $PX$ (resp. $QX$) denote the tangent (resp. normal) component of $JX$.

We now denote by $\nabla$ (resp. $\nabla^\perp$) the Riemannian connection in $\bar{M}$ (resp. $M$) with respect to the Riemannian metric $g$. The linear connection induced by $\nabla$ on the normal bundle $T^\perp M$ is denoted by $\nabla^\perp$. Thus the Gauss and Weingarten formulas are given by

$$\nabla_X Y = \nabla_X Y + h(X, Y), \quad \nabla_X N = -A_N X + \nabla_X N^\perp$$

for all $X, Y \in TM$ and $N \in T^\perp M$, where $h$ is the second fundamental form of $M$ and $A_N$ is the fundamental tensor with respect to the normal section $N$. These tensor fields are related by

$$g(h(X, Y), N) = g(A_N X, Y).$$

The equation of Gauss is given by

$$R(X, Y, z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)).$$

DEFINITION. A CR-submanifold $M$ is called $D$-minimal (resp. $D^\perp$-minimal) if $P h(E_i, E_i) = 0$ (resp. $E_i^\perp h(F_i, F_i) = 0$) where $\{E_1, E_2, \ldots, E_p\}$ (resp. $\{F_1, F_2, \ldots, F_q\}$ is a local field of frames of $D$ (resp. $D^\perp$).

DEFINITION. A CR-submanifold $M$ is called $D$-totally geodesic (resp. $D^\perp$-totally geodesic) if $h(X, Y) = 0$ for each $X, Y \in D$ (resp. $X, Y \in D^\perp$). $M$ is called a mixed totally geodesic if $h(X, Z) = 0$ for each $X \in D, Z \in D^\perp$.


Let $M$ be a 3-dimensional CR-submanifold of $S^6$. It is known that $S^6$ is nearly Kaehler manifold of constant type 1. Suppose $\dim D = 2, \dim D^\perp = 1$, and $\{E_1, JE_1\}$ be a local frame in $D$ and $\{F\}$ that of $D^\perp$.

The mean curvature vector $H$ is defined by

$$H = \frac{1}{3} \left\{ \sum_{i=1}^{2p} h(E_i, E_i) + h(F, F) \right\}.$$ (3.1)

If $H = 0$, then $M$ is said to be minimal. Now we define

$$H_D = \frac{1}{2} \sum_{i=1}^{2p} h(E_i, E_i), H_{D^\perp} = h(F, F).$$ (3.2)

If $H_D = 0$, then $M$ is said to be $D$-minimal and if $H_{D^\perp} = 0$, then $M$ is said to be $D^\perp$-minimal.

Let $U, V$ be any vector field tangent to CR-submanifold $M$. The Ricci tensor and the scalar curvature are respectively given by

$$S(U, V) = \sum_{i=1}^{2p} g(R(E_i, U)V, E_i) + g(R(F, U)V, F),$$ (3.3)

$$\rho = \sum_{i=1}^{2p} S(E_i, E_i) + S(F, F).$$ (3.4)
Also
\[ S_D(U,V) = g(R(E_i,U)V,E_i), S_D(U,V) = g(R(F,U)V,F). \] (3.5)

\[ \rho_{DD} = \sum_{i=1}^{2} S_D(E_i,E_i), \rho_{DD} = S_D(F,F). \] (3.6)

\[ \rho_{D} = \sum_{i=1}^{2} S_D(E_i,E_i), \rho_{D} = S_D(F,F). \] (3.7)

Now using (2.2) and (2.6), we have for \( X,Y \in TM \)
\[ S_D(X,Y) = 2g(X,Y) - g(PX, PY) + 2g(HD, h(X,Y)) \] (3.8)

\[ S_D(X,Y) = g(X,Y) - g(QX, QY) + g(HD, h(X,Y)) \] (3.9)

\[ \rho_{DD} = 2 + 4g(HD, HD) - \sum_{i,J=1}^{2} \| h(E_i,F) \|^2, \] (3.10)

\[ \rho_{DD} = 2 + 2g(HD, HD) - \sum_{i=1}^{2} \| h(E_i,F) \|^2, \] (3.11)

\[ \rho_{D} = g(HD, HD) - \| h(F,F) \|^2. \] (3.12)

It is easy to see that
\[ \rho_{DD} = \rho_{D}. \]

Now we prove

**THEOREM 1.** Let \( M \) be a \( D \)-minimal CR-submanifold of a 6-dimensional sphere \( S^6 \). Then the following hold:

(a) \( S_D(X,X) - 2 \| X \|^2 + \| PX \|^2 \leq 0 \), for \( X \in TM \)

(b) \( \rho_{DD} \leq 2 \)

(b') \( \rho_{DD} \leq 2. \)

The equality in (a) for \( X \in D \), and in (b) holds if and only if \( M \) is \( D \)-totally geodesic.

The equality in (a) for \( X \in D^\perp \) and in (b') holds if and only if \( M \) is mixed totally geodesic.

**PROOF.** Since \( M \) is \( D \)-minimal, from (3.8), we have
\[ S_D(X,X) - 2 \| X \|^2 + \| PX \|^2 = \sum_{i=1}^{2} g(h(E_i,X),h(E_i,X)). \]

This proves (a) and (b), (b') follow from (3.10) and (3.11). Similarly, we have

**THEOREM 2.** Let \( M \) be a \( D^\perp \)-minimal CR-submanifold of a 6-dimensional sphere \( S^6 \). Then the following hold:

(a) \( S_D^\perp(X,X) - \| X \|^2 + \| QX \|^2 \leq 0 \), for \( X \in TM \)

(b) \( \rho_{D} \leq 2. \)
The equality for $X \in D^\perp$ in (a) and (b') holds if and only if $M$ is $D^\perp$-totally geodesic.

The equality for $X \in D$ in (a) and in (b) holds if and only if $M$ is mixed totally geodesic.

**Proof.** Since $M$ is $D^\perp$-minimal, so from (3.9), we have

$$S_{D^\perp}(X,X) = \|X\|^2 + \|QX\|^2 = -g(h(F,X),h(F,X)).$$

which proves (a) and (b), (b') follows from (3.11) and (3.12).

**Remarks.** The example given by Sekigawa [6] is an example of $D$-totally geodesic and $D^\perp$-totally geodesic (and hence minimal) proper CR-submanifold of a 6-dimensional sphere and this illustrates the Theorem in the sense that $S^3 \times S^1$, where $f$ is a function on $S^3$, is a $D$-minimal CR-submanifold of $S^6$ in which it is easily verified that $\rho_{DD} = 2$. The equality arises because it is also $D$-totally geodesic in $S^6$.

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**References**

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