FACTORIAL RATIOS THAT ARE INTEGERS

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1. INTRODUCTION.

The expressions

$$\frac{(2n)!}{n!(n+1)!}, \quad (1.1)$$

$$\frac{(2r+1)!}{r!} \cdot \frac{(2n)!}{n!(n+r+1)!}, \quad (1.2)$$

$$s \cdot \frac{(2n+s-1)!}{n!(n+s)!}, \quad (1.3)$$

$$\frac{(s+2r)!}{(s-1)!r!} \cdot \frac{(2n+s-1)!}{n!(n+s+r)!}, \quad (1.4)$$

are always integers. They are called the Catalan, generalized Catalan, ballot, and the super ballot numbers, respectively [1]. Here we consider two results concerning divisibility by expressions involving factorials, which generalize these and other similar assertions.

For given positive integers $a_1, a_2, \dots, a_t$, let $\{a_1, a_2, \dots, a_t\}$ denote the least common multiple of these integers. For integers n and k , $n > k \geq 0$, set

$$L(n, k) = \{n, n-1, \dots, n-k\}. \quad (1.5)$$

The novel aspect of our approach is the introduction of the function

$$Q(J, B, C) = \prod_{i=0}^J (B-i, L(C, i)), \quad (1.6)$$

for $B \geq C > J \geq 0$, where (α, β) denotes the greatest common divisor of the integers α and β .

Our results describe divisibility properties of this function "from above" and "from below". We have

THEOREM 1.1. let m, k, J , be positive integers such that $m \geq k > J \geq 0$, then the number $F(J, m, k)$ given by

$$F(J, m, k) = \frac{Q(J, m, k)}{m(m-1) \dots (m-J)} \cdot \binom{m}{k} \quad (1.7)$$

is always an integer, where $\binom{m}{k}$ is the binomial coefficient.

THEOREM 1.2. For integers $s \geq 1$, $r \geq 0$, and $n \geq 1$, the integer

$$P(r, s) = \frac{(2r+s)!}{r!(s-1)!} \quad (1.8)$$

is a multiple of

$$Q(r, r + s + 2n, r + s + n). \quad (1.9)$$

Applying Theorem 1.1 with $J = 0$, gives that for $m \geq k > 0$,

$$\frac{(m, k)}{m} \binom{m}{k} \quad (1.10)$$

is an integer. (Note that (1.10) holds also for $k = 0$.) Taking $m = 2n + s$, $k = n$, (1.10) yields that

$$\frac{(2n + s, n)}{2n + s} \binom{2n + s}{n} = (2n + s, n) \cdot \frac{(2n + s - 1)!}{n!(n + s)!}$$

is an integer. Since $(2n + s, n) = (s, n)$ divides s , we have that (1.3) is an integer. Then (1.1) is the special case $s = 1$.

As for the expression (1.4), we apply Theorem 1.2, with $s \geq 1$, $r \geq 0$ and $n \geq 1$, obtaining that $P(r, s)$ is a multiple of $Q(r, r + s + 2n, r + s + n)$. But by Theorem 1.1,

$$\begin{aligned} & \frac{Q(r, r + s + 2n, r + s + n)}{(r + s + 2n)(r + s + 2n - 1) \dots (s + 2n)} \cdot \binom{r + s + 2n}{r + s + n} \\ &= Q(r, r + s + 2n, r + s + n) \cdot \frac{(s + 2n - 1)!}{n!(r + s + n)!} \end{aligned}$$

is an integer. Thus (1.4) is an integer. Then (1.2) is the special case $s = 1$.

2. PROOF OF THEOREM 1.1.

If not specified otherwise, all letters denote positive integers. Suppose that an integer X is given as a product:

$$X = \prod_{i=1}^f X_i. \quad (2.1)$$

For any positive integer A we define

$$N(A, X) = \text{the number of } X_i \text{ divisible by } A. \quad (2.2)$$

In all applications of this notation, the reference product (2.1) will be uniquely given. For any prime p , let

$$\text{Pow}(p, X) = \text{the largest } \alpha \text{ such that } p^\alpha \text{ divides } X \quad (2.3)$$

It is easy to see that

$$\text{Pow}(p, X) = \sum_{\tau=1}^{\infty} N(p^\tau, X). \quad (2.4)$$

The following two lemmas are clear.

LEMMA 2.1. If X is given by (2.1) and $Y = \prod_{j=1}^h Y_j$ is, such that, for all primes p and $\tau > 0$, we have

$$N(p^\tau, Y) \geq N(p^\tau, X), \quad (2.5)$$

then X divides Y .

LEMMA 2.2. For $n \geq 1$, let $n! = \prod_{j=1}^n j$ be the reference product for $n!$. Then

$$N(p^\tau, n!) = \left\lfloor \frac{n}{p^\tau} \right\rfloor, \quad (2.6)$$

where $[a]$ is the number of positive integers $\leq a$.

From (1.7) we have

$$F(J, m, k) = Q(J, m, k) \frac{(m - J - 1)!}{k!(m - k)!}. \quad (2.7)$$

Write $Q(J, m, k)$ in the form (2.1):

$$Q(J, m, k) = \prod_{i=1}^J (m - i, L(k, i)) = \prod_{i=0}^J Q_i(m, k). \quad (2.8)$$

By Lemmas 2.1 and 2.2, it is enough to show that

$$N(p^r, Q) + \left\lfloor \frac{m - J - 1}{p^r} \right\rfloor \geq \left\lfloor \frac{m - k}{p^r} \right\rfloor + \left\lfloor \frac{k}{p^r} \right\rfloor. \quad (2.9)$$

Set

$$\Delta(p^r, F) = N(p^r, Q) + \left\lfloor \frac{m - J - 1}{p^r} \right\rfloor - \left\lfloor \frac{m - k}{p^r} \right\rfloor - \left\lfloor \frac{k}{p^r} \right\rfloor, \quad (2.10)$$

so that (2.9) is equivalent to

$$\Delta(p^r, F) \geq 0. \quad (2.11)$$

Let

$$\frac{m - k}{p^r} = \left\lfloor \frac{m - k}{p^r} \right\rfloor + \frac{d_r}{p^r}, \quad \frac{k}{p^r} = \left\lfloor \frac{k}{p^r} \right\rfloor + \frac{e_r}{p^r}, \quad (2.12)$$

where,

$$0 \leq d_r \leq p^r - 1, \quad 0 \leq e_r \leq p^r - 1. \quad (2.13)$$

Then

$$\frac{m - J - 1}{p^r} = \left\lfloor \frac{m - k}{p^r} \right\rfloor + \left\lfloor \frac{k}{p^r} \right\rfloor + \frac{d_r + e_r - J - 1}{p^r}, \quad (2.14)$$

implying

$$\left\lfloor \frac{m - J - 1}{p^r} \right\rfloor = \left\lfloor \frac{m - k}{p^r} \right\rfloor + \left\lfloor \frac{k}{p^r} \right\rfloor + \left\lfloor \frac{d_r + e_r - J - 1}{p^r} \right\rfloor \quad (2.15)$$

From (2.10) and (2.15) we have:

$$\Delta(p^r, F) = N(p^r, Q) + \left\lfloor \frac{d_r + e_r - J - 1}{p^r} \right\rfloor. \quad (2.16)$$

If $d_r + e_r - J - 1 \geq 0$, then $\Delta(p^r, F) \geq 0$. Suppose that $d_r + e_r - J - 1 < 0$, then $d_r + e_r \leq J$.
If

$$L = d_r + e_r, \quad (2.17)$$

$0 \leq L \leq J$. By (2.12) we have that p^r divides both $m - k - d_r$ and $k - e_r$, and hence it divides $m - (d_r + e_r) = m - L$. Then p^r divides $(m - L, k - e_r)$. For $t \geq 0$ we have

$$p^r \mid (m - L - tp^r, k - e_r). \quad (2.18)$$

For each t such that $L + tp^r \leq J$, p^r divides:

$$(m - L - tp^r, \{k, k - 1, \dots, k - e_r, \dots, k - L - tp^r\}) = Q_{L+tp^r}(m, k).$$

Thus each $0 \leq t \leq \left\lfloor \frac{J-L}{p^r} \right\rfloor$ maps onto $Q_{L+tp^r}(m, k)$ that is divisible by p^r . Since this map is 1-1 into the factors $Q_i(m, k)$ in (2.8) that are divisible by p^r , it follows that

$$N(p^r, Q) \geq 1 + \left\lfloor \frac{J - L}{p^r} \right\rfloor. \quad (2.19)$$

From (2.16), (2.17), and (2.19) we have

$$\Delta(p^\tau, F) \geq 1 + \left\lfloor \frac{J-L}{p^\tau} \right\rfloor + \left\lfloor \frac{L-J-1}{p^\tau} \right\rfloor. \quad (2.20)$$

It is easy to see that

$$\left\lfloor \frac{L-J-1}{p^\tau} \right\rfloor = - \left\lfloor \frac{J-L}{p^\tau} \right\rfloor - 1. \quad (2.21)$$

Since (2.20) and (2.21) imply (2.11), Theorem 1.1 is proved.

3. PROOF OF THEOREM 1.2.

LEMMA 3.1. Let

$$U = \prod_{i=1}^a U_i, \quad V = \prod_{j=1}^b V_j, \quad W = \prod_{l=1}^c W_l, \quad Z = \prod_{k=1}^d Z_k. \quad (3.1)$$

For all primes p and integers $\tau > 0$, we assume that

$$N(p^\tau, W) \leq \min(N(p^\tau, U), N(p^\tau, V)), \quad (3.2)$$

and

$$N(p^\tau, Z) \leq \max(N(p^\tau, U), N(p^\tau, V)). \quad (3.3)$$

Then $\frac{UV}{W}$ is an integer divisible by Z .

PROOF. We have for any prime p ,

$$\text{Pow}\left(p, \frac{UV}{W}\right) = \sum_{\tau=1}^{\infty} (N(p^\tau, U) + N(p^\tau, V) - N(p^\tau, W)). \quad (3.4)$$

Let

$$\lambda(p^\tau) = N(p^\tau, U) + N(p^\tau, V) - N(p^\tau, W).$$

Via (3.2) and (3.3) we have:

$$\begin{aligned} \lambda(p^\tau) &= \max(N(p^\tau, U), N(p^\tau, V)) + \min(N(p^\tau, U), N(p^\tau, V)) - N(p^\tau, W) \\ &\geq N(p^\tau, Z) + N(p^\tau, W) - N(p^\tau, W) = N(p^\tau, Z). \end{aligned}$$

This and (3.4) yield

$$\text{Pow}\left(p, \frac{UV}{W}\right) \geq \sum_{\tau=1}^{\infty} N(p^\tau, Z) = \text{Pow}(p, Z),$$

and the lemma follows.

Write (1.9) in the form:

$$Q(r, r+s+2n, r+s+n) = \prod_{k=0}^r Q_k, \quad (3.5)$$

where

$$Q_k = (r+s+2n-k, L(r+s+n, k)). \quad (3.6)$$

We also rewrite (1.8) in the form:

$$P(r, s) = \prod_{i=0}^{r-1} (2r+s-i) \prod_{j=0}^r (r+s-j) / r! \quad (3.7)$$

We will obtain Theorem 1.2 by applying Lemma 3.1 with

$$\begin{aligned} U &= \prod_{i=0}^{r-1} U_i = \prod_{i=0}^{r-1} (2r + s - i), \\ V &= \prod_{j=0}^r V_j = \prod_{j=0}^r (r + s - j), \\ W &= \prod_{l=0}^{r-1} W_l = \prod_{l=0}^{r-1} (l + 1) = r!, \\ Z &= \prod_{k=0}^r Z_k = \prod_{k=0}^r Q_k. \end{aligned}$$

Thus $Z = Q(r, r + s + 2n, r + s + n) = Q$, and

$$P(r, s) = \frac{UV}{W} = \binom{2r+s}{r} \frac{(r+s)!}{(s-1)!}$$

is an integer. As for (3.2) we have:

$$\begin{aligned} N(p^r, W) &= N(p^r, r!) = \left\lfloor \frac{r}{p^r} \right\rfloor, \\ N(p^r, U) &= \left\lfloor \frac{2r+s}{p^r} \right\rfloor - \left\lfloor \frac{r+s}{p^r} \right\rfloor, \\ N(p^r, V) &= \left\lfloor \frac{r+s}{p^r} \right\rfloor - \left\lfloor \frac{s-1}{p^r} \right\rfloor, \end{aligned}$$

from which the inequalities $N(p^r, W) \leq N(p^r, U)$, $N(p^r, W) \leq N(p^r, V)$ are obvious. Thus the proof reduces to establishing (3.3). Consider those Q_k , $0 \leq k \leq r$, such that p^r divides Q_k . Since this requires that p^r divides

$$L(r + s + n, k) = \{r + s + n, r + s + n - 1, \dots, r + s + n - k\},$$

the smallest k for which this occurs is μ^* , where

$$r + s + n - \mu^* \equiv 0 \pmod{p^r}, \quad 0 \leq \mu^* < p^r, \quad \mu^* \leq r. \quad (3.8)$$

(It is the last inequality that constrains, in part, the existence of such a Q_k .) Also, there would be a smallest k^* , $\mu^* \leq k^* \leq r$, such that

$$r + s + 2n - k^* \equiv 0 \pmod{p^r}. \quad (3.9)$$

From (3.8) and (3.9) we have

$$n \equiv k^* - \mu^* \pmod{p^r}. \quad (3.10)$$

Thus (3.9) is equivalent to

$$r + s + 2(k^* - \mu^*) - k^* \equiv 0 \pmod{p^r}, \quad \mu^* \leq k^* \leq r. \quad (3.11)$$

If (3.8) and (3.11) are not satisfied then $N(p^r, Z) = N(p^r, Q) = 0$, and (3.3) certainly holds. Thus we may assume that $N(p^r, Q) > 0$. The integers k such that p^r divides Q_k are precisely those such that

$$k^* \leq k \leq r, \quad k \equiv k^* \pmod{p^r}, \quad (3.12)$$

which gives

$$N(p^r, Q) = 1 + \left\lfloor \frac{r - k^*}{p^r} \right\rfloor. \quad (3.13)$$

Consider two cases:

Case I. $\mu^* \leq k^* \leq 2\mu^*$. Here, for all k satisfying (3.12), we have

$$r + s + 2(k^* - \mu^*) - k \leq r + s + 2(k^* - \mu^*) - k^* \leq r + s,$$

and

$$r + s + 2(k^* - \mu^*) - k \geq r + s + 2(k^* - \mu^*) - r \geq s.$$

Note that this implies

$$r \geq k - 2(k^* - \mu^*) \geq 0.$$

Thus, in this case, a factor Q_k which is divisible by p^r maps onto $V_{k-2(k^*-\mu^*)}$, which is divisible by p^r . Since this map is 1-1 into the set of V_j that are divisible by p^r , we have

$$N(p^r, V) \leq N(p^r, Q). \quad (3.14)$$

Case II. $k^* > 2\mu^*$. Let

$$q^* = r + s + k^* - 2\mu^*. \quad (3.15)$$

By (3.11), we have $q^* \equiv 0 \pmod{p^r}$. Also

$$q^* \geq r + s + 1,$$

and

$$q^* \leq r + s + k^* \leq 2r + s.$$

Thus q^* is one of the U_i and is divisible by p^r . Hence the integers of the form $q^* + tp^r$ such that

$$q^* + tp^r \leq 2r + s, \quad t \geq 0, \quad (3.16)$$

are also among the U_i 's, which are divisible by p^r . This yields

$$N(p^r, U) \geq 1 + \left\lfloor \frac{2r + s - q^*}{p^r} \right\rfloor, \quad (3.17)$$

or inserting (3.15),

$$N(p^r, U) \geq 1 + \left\lfloor \frac{r - k^* + 2\mu^*}{p^r} \right\rfloor. \quad (3.18)$$

(Actually equality can be proved in (3.18), but this is not needed). Via (3.13) and (3.18), the inequality

$$N(p^r, Q) \leq N(p^r, U)$$

would be a consequence of

$$\left\lfloor \frac{r - k^*}{p^r} \right\rfloor \leq \left\lfloor \frac{r - k^* + 2\mu^*}{p^r} \right\rfloor.$$

But the last inequality is obvious since $\mu^* \geq 0$, and the theorem follows.

REFERENCES

1. Ira M. Gessel, Super Ballot Numbers, Brandeis University, Preprint.
2. Richard K. Guy, Unsolved Problems in Number Theory, Springer Verlag, NY, 1981, pp. 49.