A NOTE ON QUASI AND BI-IDEALS IN TERNARY SEMIGROUPS

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ABSTRACT. In this paper we have studied the properties of Quasi-ideals and Bi-ideals in ternary semi groups. We prove that every quasi-ideal is a bi-ideal in T but the converse is not true in general by giving several example in different context .

KEY WORDS AND PHRASES. Quasi-ideal, Bi-ideal, Ternary Semi group.

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1. INTRODUCTION.

D.H. Lehmer [4] gave the definition of a ternary semi group as follows:

DEFINITION 1.1. A non-empty set T is called a ternary semigroup if a ternary operation [] on T is defined and satisfies the associative law

$$[(x_1 \ x_2 \ x_3 | x_4 \ x_5] = [x_1 [x_2 \ x_3 \ x_4 | x_5] = [x_1 \ x_2 [x_3 \ x_4 \ x_5]]$$

for all $x_i \in T$, $1 \leq i \leq 5$.

Banach showed by an example that a ternary semi group does not necessarily reduce to an ordinary semi group. This has been shown by the following example.

EXAMPLE 1.2. Let $T = \{-i, 0, i\}$ be a ternary semi group under the multiplication over complex number while T is not a binary semi group under the multiplication over complex number.

Los [5] showed that any ternary semi group however may be embedded in an ordinary semi group in such a way that the operation in the ternary semi group is an (ternary) extension of the (binary) operation of the containing semi group.

Dudek [1], Feizullaer [2], Kim and Roush [3], Lyapin [6] and Sioson [7] has also studied the properties of the ternary semi groups.

We give the following definitions of ideals [7] as follows:

DEFINITION 1.3. A left (right, lateral) ideal of a ternary semi group T is a non-empty subset L(R,M) of T such that

$$[TTL] \subseteq L([RTT] \subseteq R, [TMT] \subseteq M)$$

DEFINITION 1.4. If a non-empty subset of T is a left, right and lateral ideal of T , then it is called an ideal of T .

DEFINITION 1.5. For each element t in T , the left, right and lateral ideal generated by ' t ' are respectively given by:

$$(t)_L = \{t\} \cup [Tt]$$

$$(t)_R = \{t\} \cup [tT]$$

$$(t)_M = \{t\} \cup [TtT] \cup [TTtTT]$$

Due to associative law in T , one may write Sioson [7]

$$\begin{aligned} [x_1 x_2 \dots x_{2n+1}] &= [x_1 \dots x_m x_{m+1} \dots x_{m+4} \dots x_{2n+1}], m \leq n \\ &= [x_1 \dots [x_m x_{m+1} x_{m+2}] x_{m+3} x_{m+4} \dots x_{2n+1}], m \leq n \end{aligned}$$

DEFINITION 1.6. Quasi-ideal in a ternary semi group [7] is also a subset Q of T (possibly empty) satisfying following two conditions:

- (1) $[QTT] \cap [TQT] \cap [TTQ] \subseteq Q$
- (2) $[QTT] \cap [TTQTT] \cap [TTQ] \subseteq Q$

REMARK 1.7. Every right, left and lateral ideal is a quasi-ideal. But every quasi-ideal is not a right, a left and a lateral ideal of T . This follows from the following example

EXAMPLE 1.8. Let $T = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$ be the ternary semi group under matrix multiplication. Then $Q = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \}$ be the quasi-ideal of T , which is neither a left, nor a right nor a lateral ideal of T .

DEFINITION 1.9. A ternary sub-semi group is a subset S of a ternary semi group T such that

$$[SSS] \subseteq S$$

DEFINITION 1.10. A ternary semi group T is said to be a ternary group if it satisfies the following property that for all x, y and z in T , there exists unique a, b, c in T such that

$$[xab] = c, [ayb] = c, [abz] = c$$

DEFINITION 1.11. A ternary group T is said to be a ternary group with 0 if for all a, b, c in T

$$[oab] = 0 = [aob] = [abo] = [aoo] = [obo] = [ooc].$$

DEFINITION 1.12. A ternary semi group T is with identity if there exists an

idempotent e in T such that

$$[aee] = [eaa] = [aea] = a, \quad \forall a \in T.$$

2. SOME RESULTS ON QUASI-IDEAL IN T WHICH ARE TRIVIAALLY TRUE

PROPOSITION 2.1. A ternary group T with 0 and $[TTT] \neq 0$ has no proper quasi-ideal.

PROPOSITION 2.2. The intersection of a quasi-ideal Q and a ternary sub semi-group A of a ternary semi group T is either empty or a quasi-ideal of A .

PROPOSITION 2.3. Let Q be any non-empty subset of a ternary semi group T , then the following are true:

- (1) $Q \cup [TTQ]$ is the smallest left ideal of T containing Q .
- (2) $Q \cup [QTT]$ is the smallest right ideal of T containing Q .
- (3) $Q \cup [TQT] \cup [TTQTT]$ is the smallest lateral ideal of T containing Q .
- (4) If Q is a quasi-ideal of T . Then

$$Q = (Q \cup [TTQ]) \cap (Q \cup [TQT] \cup [TTQTT]) \cap (Q \cup [QTT]).$$

PROPOSITION 2.4. The intersection of arbitrary set of quasi-ideals in a ternary semi group is either empty or a quasi-ideal of T .

DEFINITION 2.5. Let X be a non-empty subset of a ternary semi group T . The quasi-ideal of T generated by X is intersection of all quasi-ideals $(X)_Q$ of T containing X .

If the subset X consists of a single element x , then $(x)_Q$ is the cyclic quasi-ideal of T .

PROPOSITION 2.6. Let X be a non-empty subset of ternary semi group T , then

$$(X)_Q = (X \cup [TTX]) \cap (X \cup [TXT] \cup [TTXTT]) \cap (X \cup [XTT])$$

is the smallest quasi-ideal containing X .

PROOF. Sioson [7] shows that the intersection of a right, a left and a lateral ideal of a ternary semi group T is a quasi-ideal. Therefore the proof easily follows by using 2.3.

From 2.6 it follows that

$$(x)_Q = (\{x\} \cup [TTx]) \cap (\{x\} \cup [TxT] \cup [TTxTT]) \cap (\{x\} \cup [xTT])$$

is the smallest quasi-ideal of T containing x .

3. BI-IDEALS IN TERNARY SEMI GROUP

DEFINITION 3.1. A ternary sub semi group B of a ternary semi group T is a bi-ideal of T if $[BTBTB] \subseteq B$.

PROPOSITION 3.2. Every quasi-ideal of a ternary semi group T is a bi-ideal.

PROOF. Let Q be a quasi-ideal of T . Then Q is a ternary semi group of T .
 Now $[QTQTQ] \subseteq [Q(TTT)T] \subseteq [QTT]$.
 Similarly $[QTQTQ] \subseteq [TTQ] \cap [TTQTT]$.
 Therefore $[QTQTQ] \subseteq [TTQ] \cap [TTQTT] \cap [QTT] \subseteq Q$.

PROPOSITION 3.3. Let A be an ideal and Q be a quasi-ideal of T . Then $A \cap Q$ is a bi-ideal and a quasi-ideal of T .

PROOF. $[A \cap Q \ A \cap Q \ A \cap Q] \subseteq [AAA] \cap [QQQ] \subseteq A \cap Q$ implies that $A \cap Q$ is a ternary sub semi group of T . Also

$$[A \cap Q \ T \ A \cap Q \ T \ A \cap Q] \subseteq [QTQTQ] \cap [A(TAT)A] \subseteq Q \cap [AAA]$$

by (3.2) and the given hypothesis implies that L.H.S. $\subseteq Q \cap A$. Thus $A \cap Q$ is a bi-ideal of T . Since A is an ideal of T and it is also a quasi-ideal of T . Hence $A \cap Q$ is a quasi-ideal of T .

PROPOSITION 3.4. Let X, Y be non-empty subsets of ternary semi group T , then $N = [XTY]$ is a bi-ideal of T .

PROOF. Clearly N is a ternary sub semi group of T . Also

$$\begin{aligned} [NININ] &\subseteq [X(TTT)(TTT)(TTT)Y] \subseteq [X(TTT)Y] \\ &\subseteq [XTY] = N. \end{aligned}$$

Then N is a bi-ideal of T .

PROPOSITION 3.5. The intersection of arbitrary set of bi-ideals of T is either empty or a bi-ideal of T .

We omit the trivial proof.

PROPOSITION 3.6. Every left, right or lateral ideal of T is a bi-ideal of T .

PROOF. Trivial.

PROPOSITION 3.7. Let Q be a subset of a ternary semi group T and Y be a non-empty proper subset of T such that

- (1) $[TTQ] \cup [TQT] \cup [QTT] \cup [TTQTT] \subseteq Y$.
- (2) $Y \subseteq Q$.

Then Y is an ideal of T . Moreover Y is a bi-ideal of T .

PROOF. It is obvious that $[TTY]$, $[TYT]$, $[YTT]$ and $[TTYTT]$ are contained in Y under the condition (2) therefore Y is an ideal of T . And hence a quasi-ideal of T which by 3.2 is a bi-ideal of T .

In the following example we show that if both or either of the conditions (1) and (2) of above proposition are not satisfied then Y is neither a left, a right, a lateral, a quasi nor a bi-ideal of T .

EXAMPLE 3.8. Let $T = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$.

Then T is a ternary semi group under matrix multiplication.

(1) Take $Y = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \}$ and the quasi-ideal

$$Q = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \} \text{ of } T.$$

We see that $Y \not\subseteq Q$, and

$$\begin{aligned} [TTQ] \cup [TQT] \cup [TTQTT] \cup [QTT] \\ = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \} \\ \not\subseteq Y \end{aligned}$$

Also each of $[TTY]$, $[TYT]$, $[YTT]$ and $[TTYTT]$ is not in Y .

Therefore Y is neither a left, nor a right nor a lateral ideal of T .

Moreover $[TTY] \cap [TYT] \cap [YTT] \not\subseteq Y$.

So, Y is not a quasi-ideal of T .

(2) Take $Y = \{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \}$ and $Q = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \}$. Then $Y \subseteq Q$.

Again $[TTQ] \cup [TQT] \cup [QTT] \cup [TTQTT] \not\subseteq Y$.

Since each of $[TTY]$, $[TYT]$, $[YTT]$ contains $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, they are not contained in Y .

Hence Y is neither a left, a lateral nor a right ideal of T .

Also $[TTY] \cap [TYT] \cap [YTT] \not\subseteq Y$.

So Y is not a quasi-ideal of T .

Further $[YTYTY] \not\subseteq Y$ implies Y is not a bi-ideal of T .

(3) Now we take $Y = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \}$ and $Q = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \}$ of T . Then $Y \not\subseteq Q$.

$$[TTQ] \cup [TQT] \cup [TTQTT] \cup [QTT] = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \} \subseteq Y.$$

We find that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in [TTY]$, $[TYT]$ and $[YTT]$.

But $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \notin Y$. So Y is either a left, a lateral nor a right ideal of T . Similarly

Y is neither a quasi nor a bi-ideal of T .

THEOREM 3.9. Let X, Y and Z be three non-empty subsets of a ternary semi-group T and $N = [XYZ]$. Then N is a bi-ideal of T if one of the following conditions holds:

- (1) $X, Y \subseteq Z$ and Z is a bi-ideal of T .
- (2) $Y, Z \subseteq X$ and X is a bi-ideal of T .
- (3) $X, Z \subseteq Y$ and Y is a bi-ideal of T .
- (4) At least one of X, Y, Z is a right, or a left or a lateral ideal of T .

PROOF. (1) $[NNN] \subseteq [XYZ[ZZZ][ZZZ]]$

$$\subseteq [XY[ZZZ]] \subseteq N$$

and $[NNTN] \subseteq [XY[ZZTZ]] \subseteq N$.

Similar proofs establish (2) and (3).

(4) Assume X is a right ideal of T . Then

$$[NNN] \subseteq [X[TTT][TIT]YZ] \subseteq [XTITYZ] \subseteq N$$

$$[NNTN] \subseteq [X[TTT][TIT]TIZ] \subseteq [XTITYZ] \subseteq N.$$

Similar proofs can be given when either X or Y or Z is a left, or a lateral or a right ideal of T.

DEFINITION 3.10 [7]. An element 't' in a ternary semi group T is said to be regular if there exists x,y in T such that

$$[txtyt] = t.$$

If all the elements of T are regular then it is said to be regular ternary semi group.

EXAMPLE 3.11. This example shows that there exists a ternary semi group while T is not a regular ternary semi group such that T has a minimal right, a minimal lateral and a minimal left ideal of T.

Let $T = \{0, e, a, b\}$ be the ternary semi group under the operation (), (given below in the table)

()	0	e	a	b
0	0	0	0	0
e	0	e	a	b
a	0	a	0	0
b	0	b	0	0

$$\forall a, b, c \in T, [abc] = a(bc) = (ab)c.$$

Hence $\{0\}$ is a minimal right, a minimal left and a minimal lateral ideal of T.

Since a and b are not the regular elements of T. Therefore T is not a regular ternary semi group.

Now we use theorem 3.9 to give an example of a ternary semi group in which a bi-ideal is not a quasi-ideal.

EXAMPLE 3.12. Let T be a ternary semi group such that T is not regular, X,Y,Z be respectively a minimal right, a minimal lateral and a minimal left ideal of T satisfying the condition of 3.9. Thus $N = [XYZ]$ is a bi-ideal of T. We will show that N is not a quasi-ideal of T.

PROOF. $[XYZ] \subseteq [XTT] \subseteq X$, $[XYZ] \subseteq Y$, $[XYZ] \subseteq Z$. So, $[XYZ] \subseteq X \cap Y \cap Z$ which is a minimal quasi-ideal of T [7].

If we assume that $[XYZ]$ is a quasi-ideal then $[XYZ] = X \cap Y \cap Z$ which (by Sioson [7]) thus implies that T is a regular ternary semi group. Hence it contradicts the hypothesis. So $[XYZ]$ is not a quasi-ideal but bi-ideal by Theorem 3.9.

PROPOSITION 3.13. In a regular ternary semi group every bi-ideal is a quasi-ideal.

PROOF. Sioson [7] shows that a subset Q of a regular ternary semi group T is a quasi-ideal if and only if

$$[QTQTQ] \cap [QTTQTQ] \subseteq Q.$$

Since a bi-ideal of T , clearly satisfies the above condition, so we get the proof.

PROPOSITION 3.14. Let C be a non-empty subset of a ternary semi group T without identity. Then $C \cup \{CCC\} \cup \{CTCTC\}$ is the smallest bi-ideal of T containing C .

PROOF. Let x be any elemnt of $C \cup \{CCC\} \cup \{CTCTC\}$. Then either $x = x_1$ for x_1 in C or $x = [c_1 c_2 c_3] \in \{CCC\}$ for all c_i in C , $i = 1, 2, 3$ or $x = [c_1 t_1 c_2 t_2 c_3] \in \{CTCTC\}$ for all c_i in C , $i = 1, 2, 3$, t_i in T , $i = 1, 2$.

We will consider the elements of $\{CTCTC\}$. The other two cases will be done in similar manner. Let $x, y, z \in \{CTCTC\}$.

i.e., $x = [c_1 t_1 c_2 t_2 c_3]$, $y = [c_4 t_3 c_5 t_4 c_6]$, $z = [c_7 t_5 c_8 t_6 c_9]$, $c_i \in C$, $\forall i = 1, 2, \dots, 9$, $t_i \in T$, $\forall i = 1, 2, \dots, 6$.

Then

$$\begin{aligned} [xyz] &= [[c_1 t_1 c_2 t_2 c_3][c_4 t_3 c_5 t_4 c_6][c_7 t_5 c_8 t_6 c_9]] \\ &= [c_1 [[t_1 c_2 t_2][c_3 c_4 t_3][c_5 t_4 c_6]]c_7 [t_5 c_8 t_6]c_9] \\ &= [c_1 t_7 c_7 t_8 c_9] \text{ where} \\ t_7 &= [[t_1 c_2 t_2][c_3 c_4 t_3][c_5 t_4 c_6]] \\ t_8 &= [t_5 c_8 t_6] \end{aligned}$$

so $[xyz] \in C \cup \{CCC\} \cup \{CTCTC\}$.

Further, $[xt_9 y t_{10} z] = [[c_1 t_1 c_2 t_2 c_3]t_9 [c_4 t_3 c_5 t_4 c_6]t_{10} [c_7 t_5 c_8 t_6 c_9]]$

$$\begin{aligned} &= [c_1 [[t_1 c_2 t_2][c_3 t_9 c_4][t_5 c_5 t_4]]c_6 [t_{10} c_7 [t_5 c_8 t_6]]c_9] \\ &= [c_1 t_{11} c_6 t_{12} c_9] \end{aligned}$$

where

$$t_{11} = [[t_1 c_2 t_2][c_3 t_9 c_4][t_5 c_5 t_4]]$$

$$t_{12} = [t_{10} c_7 [t_5 t_8 t_6]], t_9, t_{10} \in T.$$

Thus

$$[xt_9 y t_{10} z] \in C \cup \{CCC\} \cup \{CTCTC\}.$$

Hence $C \cup \{CCC\} \cup \{CTCTC\}$ is a bi-ideal of T containing C .

Suppose there exists a bi-ideal R of T containing C such that

$$R \subseteq C \cup \{CCC\} \cup \{CTCTC\}.$$

Then R being a bi-ideal implies that

$$R \subseteq C \cup \{CCC\} \cup \{CTCTC\} \subseteq R \cup \{RRR\} \cup \{RTRTR\} \subseteq R.$$

Thus $R = C \cup \{CCC\} \cup \{CTCTC\}$ is the smallest bi-ideal of T contain ing C .

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