ON GENERIC SUBMANIFOLDS OF A LOCALLY CONFORMAL KAHLER MANIFOLD WITH PARALLEL CANONICAL STRUCTURES

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ABSTRACT. The study of CR-submanifolds of a Kähler manifold was initiated by Bejancu [1]. Since then many papers have appeared on CR-submanifolds of a Kähler manifold. Also, it has been studied that generic submanifolds of Kähler manifolds [2] are generalisations of holomorphic submanifolds, totally real submanifolds and CR-submanifolds of Kähler manifolds. On the other hand, many examples of generic surfaces in C^2 which are not CR-submanifolds have been given by Chen [3] and this leads to the present paper where we obtain some necessary conditions for a generic submanifolds in a locally conformal Kahler manifold with four canonical strucrures, denoted by P, F, t and f , to have parallel P, F and t . We also prove that for a generic submanifold of a locally conformal Kahler manifold, F is parallel iff t is parallel.

KEY WORDS AND PHRASES. CR-submanifold, generic submanifold and locally conformal Kähler manifold.

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1. INTRODUCTION.

Let $\bar{M}$ be an almost Hermitian manifold and Ω be the fundamental

2-form given by $g(JX, Y) = \Omega(X, Y)$, g being the Hermitian metric. Then $\bar{M}$ is called a locally conformal Kähler manifold (l.c.k.) if there is a closed 1-form ω , called the Lee-form on $\bar{M}$, such that $d\Omega = \Omega \wedge \omega$, where d and $\wedge$ denote exterior derivative operator and wedge product, respectively [4]. Let $\bar{M}$ be a l.c.k. manifold. Then the vector field B (Lee field of $\bar{M}$) is defined as $g(X, B) = \omega(X)$. Now suppose that $\bar{\nabla}$ be the Levi-Civita connection of g . Then, we have [4]

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y - \frac{1}{2} \left[\omega(X)Y + \omega(Y)X - g(X, Y)B \right], \quad (1.1)$$

where $\tilde{\nabla}$ is a torsionless linear connection on $\bar{M}$ called the Weyl connection of g .

The following result is known [4].

THEOREM. The almost Hermitian manifold $\bar{M}$ is a l.c.k. manifold if and only if there is a closed 1-form ω on $\bar{M}$ such that the Weyl connection be almost complex i.e. $\tilde{\nabla}J=0$.

DEFINITION. Let M be a submanifold of a l.c.k. manifold $\bar{M}$. The holomorphic tangent spaces to M at $x \in M$ are defined as $D_x = T_x M \cap J T_x M$, where D_x is the maximal complex subspace of $T_x M$. If the dimension of D_x is constant along M and it defines a differentiable distribution on M , then M is called a generic submanifold of $\bar{M}$. We call D_x the holomorphic distribution and the orthogonal complementary distribution $D_x^\perp$ is called purely real distribution [2]. A generic submanifold M of a l.c.k. manifold is a CR-submanifold [1] if the orthogonal complementary distribution $D_x^\perp$ of D_x in $T_x M$ is totally real i.e. $J D_x^\perp \subset T_x^\perp M$, where $T_x^\perp M$ is the normal space of M at x .

Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ and let ∇ , $\tilde{\nabla}$ be covariant differentiation on M induced by $\bar{\nabla}$ and $\tilde{\nabla}$, respectively. Then Gauss and Weingarten formulae w.r.t. $\bar{\nabla}$ and $\tilde{\nabla}$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \quad (1.2)$$

$$\tilde{\nabla}_X Y = \nabla_X Y + \tilde{h}(X, Y), \quad \tilde{\nabla}_X N = -\tilde{A}_N X + \tilde{\nabla}_X^\perp N \quad (1.3)$$

for any vector fields X, Y tangent to M and N normal to M . Here $h(\tilde{h})$ is

the second fundamental form of M with respect to $\bar{\nabla}(\bar{\nabla})$ and $\nabla^{\perp}(\bar{\nabla}^{\perp})$ is the normal connection. Moreover,

$$g(A_N X, Y) = g(h(X, Y), N). \quad (1.4)$$

The transforms JX and JN of X and N by J are decomposed into tangential and normal parts as

$$JX = PX + FX, \quad (1.5)$$

$$JN = tN + fN, \quad (1.6)$$

where P and f are endomorphisms of TM and $T^{\perp}M$ and F and t are normal bundle valued 1-form on TM and tangent bundle valued 1-form on $T^{\perp}M$, respectively. For Lee vector field B tangent to M , we put

$$B_x = (B_1)_x + (B_2)_x, \quad x \in M \quad (1.7)$$

where $(B_1)_x$ and $(B_2)_x$ are the tangential and normal components of B , respectively.

The following relations hold for a generic submanifold [2].

$$t(T^{\perp}M) = D^{\perp}, \quad PD = D \quad \text{and} \quad PD^{\perp} \subseteq D. \quad (1.8)$$

2. SOME RESULTS.

We define the covariant differentiation of P, F, t and f as follows:

$$(\bar{\nabla}'_X P)Y = \nabla_X(PY) - P\nabla_X Y, \quad (2.1)$$

$$(\bar{\nabla}'_X F)Y = \nabla_X^{\perp}(FY) - F\nabla_X^{\perp} Y, \quad (2.2)$$

$$(\bar{\nabla}'_X t)N = \nabla_X tN - t\nabla_X N, \quad (2.3)$$

$$(\bar{\nabla}'_X f)N = \nabla_X^{\perp} fN - f\nabla_X^{\perp} N \quad (2.4)$$

for any vector fields X and Y tangent to M and any vector field N normal to M .

We say that P (respectively f, F or t) is parallel if $\bar{\nabla}'P=0$ (respectively $\bar{\nabla}'f=0$, $\bar{\nabla}'F=0$ or $\bar{\nabla}'t=0$).

Using (1.1), (1.2) and (1.3), we have

LEMMA 2.1. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. Then

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{2} \left[\omega(X)Y + \omega(Y)X - g(X,Y)B_1 \right], \quad (2.5)$$

$$h(X,Y) = h(X,Y) + \frac{1}{2} g(X,Y)B_2, \quad (2.6)$$

$$\bar{A}_N X = A_N X + \frac{1}{2} \omega(N)X, \quad (2.7)$$

$$\bar{\nabla}_X^\perp N = \nabla_X^\perp N - \frac{1}{2} \omega(X)N \quad (2.8)$$

for any vector fields X, Y tangent to M and any vector field N normal to M .

The following result is known [4].

LEMMA A. The holomorphic distribution D is integrable iff

$$g(\bar{h}(X, JY), FZ) = g(\bar{h}(JX, Y), FZ)$$

for any vector fields $X, Y \in D$ and $Z \in D^\perp$.

Using (2.6) in Lemma A, we have

COROLLARY 2.2. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. Then the holomorphic distribution D is integrable iff

$$g(h(X, JY) - h(JX, Y) + \Omega(X, Y)B_2, FZ) = 0$$

for any $X, Y \in D$ and $Z \in D^\perp$.

From (1.3), (1.5) and (1.6), on comparing the tangential and normal components, we obtain

LEMMA 2.3. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. Then

$$\nabla_{\tilde{X}} PY - \tilde{A}_{FY} X = P \nabla_{\tilde{X}} Y + \tilde{t}h(X, Y), \quad (2.9)$$

$$\tilde{h}(X, PY) + \nabla_{\tilde{X}}^{\perp} FY = F \nabla_{\tilde{X}} Y + f \tilde{h}(X, Y) \quad (2.10)$$

for any vector fields X, Y tangent to M .

3. GENERIC SUBMANIFOLDS WITH $\tilde{\nabla}'P=0$.

We now assume that the canonical structure P is parallel on a generic submanifold of a l.c.k. manifold $\bar{M}$ i.e., $\tilde{\nabla}'P=0$. Then from (2.5), we have

$$\nabla_{\tilde{X}} PY - P \nabla_{\tilde{X}} Y = \frac{1}{2} \left[\omega(Y)PX - \omega(PY)X + g(X, PY)B_1 - g(X, Y)PB_1 \right]$$

for all vector fields X, Y tangent to M .

Using (2.9) in the above equation, we get

$$\tilde{t}h(X, Y) + \tilde{A}_{FY} X = \frac{1}{2} \left[\omega(Y)PX - \omega(PY)X + g(X, PY)B_1 - g(X, Y)PB_1 \right] \quad (3.1)$$

which, in view of (1.4) and (1.8), give

$$\begin{aligned} 2g(\tilde{h}(X, Z), FY) &= \omega(Y)g(PX, Z) - \omega(PY)g(X, Z) \\ &\quad + g(X, PY)g(B, Z) - g(X, Y)g(PB, Z) \end{aligned} \quad (3.2)$$

for all $Z \in D$ and $X, Y, B \in TM$.

Thus, for $X, Z \in D$ and $Y \in D^{\perp}$, equation (3.2) gives us the following:

PROPOSITION 3.1. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ such that $B \in D$. If P is parallel, then the holomorphic distribution D is integrable, that is,

$$g(\tilde{h}(D, D), FD^{\perp}) = 0.$$

Similarly, for $X, Y \in D^{\perp}$ and $Z \in D$, we have, from (3.2), the following:

PROPOSITION 3.2. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ such that $B \in D^{\perp}$. If P is parallel, then

$$g(\tilde{h}(D, D^\perp), F D^\perp) = 0.$$

We now assume that the Lee field B is normal to M . Then from (2.6), (2.7), (3.1) and using the fact that $t(T^\perp M) = D^\perp$, we have

$$th(X, Y) + \frac{1}{2} g(X, Y) B_2 = -A_{FY} X - \frac{1}{2} \omega(FY) X \quad (3.3)$$

for $X \in D$ and $Y \in D^\perp$. Then, for $Z \in D$ and using (1.6) in (3.3), we get

$$h(X, Z) = -\frac{1}{2} g(X, Z) B_2$$

whence

$$h(X, JZ) - h(JX, Z) = -g(X, JZ) B_2$$

which can be written as

$$g(h(X, JZ) - h(JX, Z) + \Omega(X, Z) B_2, FY) = 0.$$

Consequently, using Corollary (2.1), we have

PROPOSITION 3.3. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ with B normal to M . If P is parallel, then the holomorphic distribution D is integrable.

4. GENERIC SUBMANIFOLDS WITH $\bar{\nabla}' F = 0$

In this section, we obtain some results assuming that F is parallel.

PROPOSITION 4.1. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. If F is parallel, then distribution D is integrable and leaf of D is totally geodesic in M .

PROOF. Since F is parallel, we have from (2.2) that for all $X, Y \in D$, $F \nabla_X Y = 0$. Consequently, $\nabla_X Y \in D$ for any $X, Y \in D$, which shows that the distribution D is integrable and leaf of D is totally geodesic in M .

PROPOSITION 4.2. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. If F is parallel, then

$$A_{fN}X + A_NPX = \frac{1}{2} \left[g(B_2, N)X - g(B_2, N)PX - \omega(X)tN \right]$$

for any vector field X tangent to M and N normal to M .

PROOF. Using $(\tilde{\nabla}_X J)Y = 0$, the equations (1.3) and (1.5), we get

$$\tilde{\nabla}_X PY + h(X, PY) - \tilde{A}_{FY}X + \tilde{\nabla}_X^\perp FY = J(\tilde{\nabla}_X Y + h(X, Y)),$$

for all X tangent to M .

Using Lemma A and the relation $\tilde{\nabla}_X^\perp N = \nabla_X^\perp N - \frac{1}{2}\omega(X)N$, we get that

$$\begin{aligned} \tilde{\nabla}_X PY - \frac{1}{2}\omega(PY)X + \frac{1}{2}g(X, PY)(B_1 + B_2) + h(X, PY) - \tilde{A}_{FY}X - \frac{1}{2}\omega(FY)X + \nabla_X^\perp FY \\ = P\nabla_X Y + F\nabla_X Y + th(X, Y) + fh(X, Y) - \frac{1}{2}\omega(Y)(PX + FX) + \frac{1}{2}g(X, Y)(B_1 + B_2) \end{aligned}$$

for any vector fields X, Y tangent to M .

Comparing the normal components on both sides of the above equation, we get

$$(\tilde{\nabla}_X' F)(Y) = fh(X, Y) - h(X, PY) + \frac{1}{2} \left[g(X, Y)B_2 - g(X, PY)B_2 - \omega(Y)FX \right]$$

Hence, for any vector field N normal to M , we get

$$\begin{aligned} g((\tilde{\nabla}_X' F)Y, N) &= -g \left(A_{fN}Y + A_NPY, X \right) \\ &+ \frac{1}{2} \left[g(X, Y)g(B_2, N) - \omega(Y)g(FX, N) - g(X, PY)g(B_2, N) \right]. \end{aligned}$$

Thus, if F is parallel, we get the result.

From the above result, we immediately get the following :

COROLLARY 4.3. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ and let $B \in D$. If P is parallel, then

(a) $A_{fN}X - A_NPX = -\frac{1}{2}\omega(X)tN$, for any vector field X tangent to M , and

(b) $g(Jh(X, Y), N) = 0$, or $g(Jh(D, D^\perp), N) = 0$ for all $X \in D^\perp$ and $Y \in D$.

5. GENERIC SUBMANIFOLDS WITH $\tilde{\nabla}t = 0$

Now, we consider generic submanifold with parallel t . Then, using $(\tilde{\nabla}_X J)N = 0$, the equations (1.3) and (1.6), we get

$$\tilde{\nabla}_X tN + h(X, tN) - \tilde{A}_{fN}X + \tilde{\nabla}_X fN = J(-\tilde{A}_N X + \tilde{\nabla}_X^\perp N),$$

for all X tangent to M and N normal to M .

Using Lemma A in the above equation, we get

$$\begin{aligned} \tilde{\nabla}_X tN - \frac{1}{2} \omega(tN)X + \frac{1}{2} g(X, tN)B_1 + h(X, tN) + \frac{1}{2} g(X, tN)B_2 \\ - \tilde{A}_{fN}X - \frac{1}{2} \omega(fN)X + \tilde{\nabla}_X^\perp fN = t\tilde{\nabla}_X^\perp N + f\tilde{\nabla}_X^\perp N - P\tilde{A}_N X \\ - F\tilde{A}_N X - \frac{1}{2} \omega(N)(PX + FX) \end{aligned}$$

for any vector field X tangent to M and N normal to M .

Comparing tangential components, we get

$$(\tilde{\nabla}_X^\perp t)N = \tilde{A}_{fN}X - P\tilde{A}_N X + \frac{1}{2} \left[\omega(JN)X - \omega(N)PX - g(X, tN)B_1 \right] \quad (5.1)$$

for any vector field X tangent to M and N normal to M .

Thus we have

PROPOSITION 5.1. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. If t is parallel, then

$$\tilde{A}_{fN}X - P\tilde{A}_N X = \frac{1}{2} \left[\omega(N)PX - \omega(JN)X + g(X, tN)B_1 \right]$$

for any vector field X tangent to M and N normal to M .

The following is immediate.

COROLLARY 5.2. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ such that $B \in D$. If t is parallel, then

(a) $A_{fN}X - PA_NX = \frac{1}{2}g(X, tN)B_1$, for any X tangent to M , and

(b) $g(Jh(X, Y), N) = 0$, or $g(Jh(D, D^\perp), N) = 0$ for any $X \in D$ and $Y \in D^\perp$

PROPOSITION 5.3. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. Then F is parallel if and only if t is parallel.

PROOF. Let t be parallel. Then, from (5.1) we have

$$\begin{aligned} g(A_{fN}X, Y) - g(PA_NX, Y) + \frac{1}{2}g(B, JN)g(X, Y) \\ - \frac{1}{2}g(X, tN)g(B_1, Y) - \frac{1}{2}g(PX, Y)g(B_2, N) = 0 \end{aligned}$$

for any vector fields X, Y tangent to M and any vector field N normal to M . The above equation becomes

$$\begin{aligned} -g(fh(X, Y), N) + g(h(X, PY), N) - \frac{1}{2}g(B_2, N)g(X, Y) \\ + \frac{1}{2}g(FX, N)g(B_1, Y) - \frac{1}{2}g(B_2, N)g(PX, Y) = 0 \end{aligned}$$

which is equivalent to

$$fh(X, Y) - h(X, PY) + \frac{1}{2} \left[g(X, Y)B_2 - g(X, PY)B_2 - \omega(Y)FX \right] = 0$$

that is,

$$\tilde{\nabla}' F = 0.$$

REMARK

1. Let $M^\perp$ be any complex submanifold in complex number space C^r and M be any purely real submanifold of dimension p in the complex number space C^p . Then it can be verified that the Riemannian product $M^\perp \times M$ is a generic submanifold in C^{r+p} satisfying $\tilde{\nabla}' P = 0$ and $\tilde{\nabla}' F = 0$.

2. It is to be noted that many examples of generic surfaces in C^2 which are not CR-submanifolds can be found in Chen's book [3].

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