DIRECT SUMS OF J-RINGS AND RADICAL RINGS

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ABSTRACT. Let R be a ring, $J(R)$ the Jacobson radical of R , and P the set of potent elements of R . We prove that if R satisfies $(*)$ given x, y in R there exist integers $m = m(x, y) > 1$ and $n = n(x, y) > 1$ such that $x^m y = x y^n$, and if each $x \in R$ is the sum of a potent element and a nilpotent element, then N and P are ideals and $R = N \oplus P$. We also prove that if R satisfies $(*)$, and if each $x \in R$ has a representation in the form $x = a + u$, where $a \in P$ and $u \in J(R)$, then P is an ideal and $R = J(R) \oplus P$.

KEY WORDS AND PHRASES. Periodic, potent, J -ring, radical ring, direct sum.
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1. INTRODUCTION.

Throughout this paper, for the ring R , $J(R)$ will denote the Jacobson radical of R , N the set of nilpotent elements of R , and P the set of potent elements of R — that is, the set of $x \in R$ for which there exists an integer $n = n(x) > 1$ such that $x^n = x$.

If $P = R$, we call R a J -ring; if $J(R) = R$, we call R a radical ring. A ring R is called periodic if for each $x \in R$ there exist distinct positive integers m, n for which $x^m = x^n$; following [2], R is called weakly periodic if each element is the sum of a potent element and a nilpotent element. It is known [1, Lemma 1] that all periodic rings are weakly periodic, but it is an open question whether weakly periodic rings must be periodic. In this paper, we consider the following condition:

$(*)$ For each $x, y \in R$, there exist integers $m = m(x, y) > 1$ and $n = n(x, y) > 1$ such that

$$x^m y = x y^n. \quad (1.1)$$

It is obvious that the above condition $(*)$ is weaker than the condition $x^{n(x)} = x$ for all $x \in R$, since there exist non- J -rings satisfying $(*)$. As an example, consider any zero ring R , i.e. $xy = 0$ for all $x, y \in R$.

2. MAIN RESULTS.

We begin with

LEMMA 1. Let R be a ring satisfying $(*)$. Then P is a subring of R .

PROOF. If $a, b \in P$, then $a = a^m$, $b = b^n$ for some integers $m > 1$, $n > 1$. Let $e_a = a^{m-1}$ and $e_b = b^{n-1}$. Then

$$\begin{aligned} ae_a &= a = e_a a \text{ and } e_a^2 = e_a, \\ be_b &= b = e_b b \text{ and } e_b^2 = e_b. \end{aligned}$$

Thus,

$$(e_a e_b - e_a e_b e_a)^2 = 0 = (e_b e_a - e_a e_b e_a)^2. \quad (2.1)$$

Let $x = e_a$ and $y = e_a e_b - e_a e_b e_a$ in (1.1). Using (2.1), we have

$$e_a e_b - e_a e_b e_a = e_a^{m_1} (e_a e_b - e_a e_b e_a) = e_a (e_a e_b - e_a e_b e_a)^{n_1} = 0$$

for some integers $m_1 > 1$ and $n_1 > 1$.

Similarly, we get $e_b e_a - e_a e_b e_a = 0$. Hence $e_a e_b = e_a e_b e_a = e_b e_a$. Let $e = e_a + e_b - e_a e_b$. Then

$$e^2 = e, ae = ea = a, \text{ and } be = eb = b. \quad (2.2)$$

Let $x = ab$ and $y = e$ in (1.1). Using (2.2), we have $ab = abe^{n_2} = (ab)^{m_2} e = (ab)^{m_2}$ for some integers $m_2 > 1$ and $n_2 > 1$. Similarly, we have $a - b = (a - b)^{m_3}$ for some integer $m_3 > 1$. Then $ab \in P$ and $a - b \in P$ as desired. The lemma is thus proved.

THEOREM 1. Let R be a weakly periodic ring satisfying $(*)$. Then N and P are ideals and $R = N \oplus P$.

PROOF. If $x, y \in R$ and $n = n(x, y) > 1$ and $m = m(x, y) > 1$ are such that $x^m y = xy^n$, then

$$x^{1+k(m-1)} y = xy^{1+k(n-1)} \quad \text{for all positive integers } k. \quad (2.3)$$

It follows that

$$au = ua = 0 \text{ for all } a \in P \text{ and } u \in N. \quad (2.4)$$

This, together with Lemma 1 and the fact that $R = P + N$, shows that P is an ideal. To complete the proof, we need only show that N is an ideal, which by (2.4) amounts to showing that N is a subring.

Let $u_1, u_2 \in N$, and let $u_1 - u_2 = b + u$ for some $b \in P$ and $u \in N$. It follows from (2.4) that $(u_1 - u_2)^2 = (u_1 - u_2)u$, and hence that $(u_1 - u_2)^{k+1} = (u_1 - u_2)^k u$ for all $k \geq 1$. It is clear from (2.3) that $(u_1 - u_2)^k u = 0$ for some k , hence $u_1 - u_2 \in N$. A similar argument shows that $u_1 u_2 \in N$.

COROLLARY 1. Let R be a periodic ring satisfying $(*)$. Then N and P are both ideals and $R = N \oplus P$.

PROOF. Evident.

COROLLARY 2. Let R be a ring in which, given $x, y \in R$, there exist distinct integers $m = m(x, y) > 1$ and $n = n(x, y) > 1$ such that $x^m y = xy^n$. Then N and P are ideals and $R = N \oplus P$.

PROOF. For all $x \in R$, by hypothesis there exist distinct integers $m = m(x) > 1$ and $n = n(x) > 1$ such that $x^m x = x x^n$. Then R is periodic. Hence N and P are ideals of R and $R = N \oplus P$ by Corollary 1.

COROLLARY 3([1],[4],and[5]). Let R be a ring in which, given $x, y \in R$, there exists an integer $n = n(x, y) > 1$ such that $x^n y = x y^n$. Then N and P are ideals and $R = N \oplus P$.

PROOF. For all $x, y \in R$, by hypothesis there exist integers $m = m(x, y) > 1$ and $n = n(x, y) > 1$ such that

$$x^n y = x y^n \text{ and } (x^n)^m y = x^n y^m.$$

Then

$$x^{mn} y = x^n y^m = x y^{m+n-1}.$$

Since the equation $mn = m + n - 1$ has no integer solutions such that $m > 1$ and $n > 1$, there exist distinct integers $s = s(x, y) > 1$ and $t = t(x, y) > 1$ such that $x^s y = x y^t$. The corollary is thus proved by Corollary 2.

COROLLARY 4. Let R be a ring in which, given $x, y \in R$, there exist integers $m = m(x, y) > 1$ and $n = n(x, y) > 1$ such that $x^m y = x y = x y^n$. Then R is commutative.

PROOF. Obviously, R is periodic. Then N and P are ideals and $R = P \oplus N$ by Corollary 1. For all $x, y \in N$, there exists an integer $m = m(x, y) > 1$ such that

$$xy = x^m y = x^{m-1} xy = x^{2m-1} y = \dots = 0.$$

Then N is a zero ring, and hence R is commutative.

REMARK. By the same process we used in proving the above results, we can prove

Let R be a ring in which, given $x, y \in R$, there exist integers $m = m(x, y) > 1$ and $n = n(x, y) > 1$ such that $xy = x^m y^n$. Then (1) N and P are ideals with $N^2 = 0$; (2) $R = N \oplus P$ and R is commutative.

THEOREM 2. Let R be a ring satisfying (*). Suppose that each $x \in R$ has a representation in the form $x = a + u$, where $a \in P$ and $u \in J(R)$. Then P is an ideal and $R = J(R) \oplus P$.

PROOF. It is clear that $J(R) \cap P = \{0\}$. Since each $x \in R$ has a representation in the form $a + u$, where $a \in P$ and $u \in J(R)$, it suffices to prove that P is an ideal of R .

If $a \in P$ and $u \in J(R)$, then $au, ua \in J(R)$. Letting $x = e_a$ and $y = au$ in (1.1), we have

$$au = e_a^m au = e_a (au)^n = (au)^n.$$

Since $au \in J(R)$ and $n > 1$, we have $au = 0$. Similarly, $ua = 0$. Then $PJ(R) = J(R)P = \{0\}$.

For all $a \in P$, $r \in R$, writing r in the form $r = r_1 + r_2$, where $r_1 \in P$, $r_2 \in J(R)$, we get $ra = (r_1 + r_2)a = r_1 a + r_2 a = r_1 a \in P$ and $ar = a(r_1 + r_2) = ar_1 + ar_2 = ar_1 \in P$. Then P is an ideal by Lemma 1. This completes the proof of Theorem 2.

We conclude with

THEOREM 3. Let R be a semisimple ring satisfying $(*)$. Then R is isomorphic to a subdirect sum of fields.

PROOF. If R is a division ring, then, for all nonzero elements x, y in R , by (1.1) we have $x^{m-1} = y^{n-1}$. Then $[x^{m-1}, y] = 0$ for all $x, y \in R$, so R is a field by a theorem of Herstein [3].

Suppose now that R is a primitive ring. Note that condition $(*)$ is inherited by all subrings and all homomorphic images of R . Note also that no complete matrix ring (D) , over a division ring $D (t > 1)$ satisfies condition $(*)$, as a consideration of $x = E_{12}$ and $y = E$ shows. Because of these facts and the structure theorem of primitive rings, we may assume that R is a division ring. Then R is a field.

If R is a semisimple ring, then R is isomorphic to a subdirect sum of primitive rings R_α each of which as a homomorphic image of R satisfies condition $(*)$, so each R_α is a field. Thus, R is isomorphic to a subdirect sum of fields.

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