GELFAND THEOREM IMPLIES
STONE REPRESENTATION THEOREM OF BOOLEAN RINGS

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ABSTRACT. Stone Theorem about representing a Boolean algebra in terms of open-closed subsets of a topological space is a consequence of the Gelfand Theorem about representing a $B^*$-algebra as the algebra of continuous functions on a compact Hausdorff space.

KEY WORDS AND PHRASES. Banach algebra, $B^*$-algebra, Boolean algebra, Boolean ring, Stone Representation Theorem.

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1. INTRODUCTION.

Gelfand Theorem in the title is the Representation Theorem for commutative normed rings with an involution, which can be stated as follows (in American terminology):

Representation Theorem of Gelfand. For each commutative (complex) $B^*$-algebra $B$, with identity, there exists a compact Hausdorff space $S$ such that $B$ is isomorphic and isometric to the algebra $C(S)$ of all continuous complex valued functions on $S$ (see, for example, first corollary to the Theorem in 26E of Loomis [3] or Theorem A in section 73 of Simmons [7]).

The isomorphism preserves also the involution, i.e. $\tilde{a}^* = \tilde{a}$, where $\tilde{a}$ is the member of $C(S)$ corresponding to $a$ (and $\tilde{a}^*$ corresponds to $a^*$).

{A $B^*$-algebra (Sec. 72 of Simmons [7]) is a Banach algebra $B$ with an involution $x \rightarrow x^*$ such that $\|x^*x\| = \|x\|^2$ for all $x \in B$}.

2. MAIN RESULT.

In the sequel we shall establish validity of the following proposition:

THEOREM. Representation Theorem of Gelfand implies Stone’s Representation Theorem for Boolean Rings (see Appendix Three of Simmons [7]):
For each Boolean ring \( \mathcal{A} \) there exists a totally disconnected compact Hausdorff space \( S \) such that \( \mathcal{A} \) is isomorphic to the Boolean ring \( \mathcal{A}(S) \) of all open-closed subsets of \( S \) (the operations of \( \mathcal{A}(S) \) are the symmetric difference \( A \Delta B = (A \cap \tilde{B}) \cup (\tilde{A} \cap B) \) and intersection \( A \cap B, A \subset S, B \subset S \)).

**PROOF.** Assume validity of Gelfand Theorem. Let \( \oplus \) and \( \odot \) be operations of \( \mathcal{A} \) and let 1 and 0 be, respectively, its multiplicative and additive identities. We shall say that a finite set \( \{a_1, \ldots, a_n\} \) of members of \( \mathcal{A} \) is a decomposition of identity if \( a_1 \oplus a_2 \oplus \cdots \oplus a_n = 1 \) (below we shall write \( \sum_{i=1}^{n} a_i = 1 \)) and \( a_i \odot a_j = 0 \) if \( i \neq j \).

For each decomposition \( \{a_1, \ldots, a_n\} \) of identity and each set \( \{\lambda_1, \ldots, \lambda_n\} \) of complex numbers consider a formal sum \( f = \sum_{i=1}^{n} \lambda_i a_i \). Let \( B' \) be the class of all such formal sums. Let us use the following notation: we shall write \( [f] = \{a_1, \ldots, a_n\} \) and \( \lambda_f = \lambda(a_i) \) if \( f = \sum_{i=1}^{n} \lambda_i a_i \) is any member of \( B' \).

Define relation \( \sim \) on \( B' \) as follows: \( f \sim g \) if \( \lambda(a) = \lambda(b) \) for any \( a \in [f] \) and \( b \in [g] \) such that \( a \odot b \neq 0 \). It is an equivalence relation. In fact, we only need to prove reflexivity.

Let \( f \sim g \) and \( g \sim h \), where \( f = \sum \lambda_i a_i, \ g = \sum \mu_j b_j \) and \( h = \sum \nu_k c_k \). Assume that \( a_i \odot c_k \neq 0 \), then from the fact that \( \sum \mu_j b_j = 1 \) we conclude that there is some integer \( j \) such that \( a_i \odot b_j \neq 0 \). Then both \( a_i \odot b_j \neq 0 \) and \( b_j \odot c_k \neq 0 \) (note that \( b_j \odot b_j = b_j \)), which implies \( \lambda_i = \mu_j = \nu_k \).

Now let us define addition, multiplication, multiplication with complex numbers and involution on \( B' \) as follows: If \( f = \sum \lambda_i a_i, \ g = \sum \mu_j b_j \) and \( \lambda \) is a scalar then

\[
\begin{align*}
f + g &= \sum_{i,j} (\lambda_i + \mu_j) a_i \odot b_j \quad (2.1) \\
fg &= \sum_{i,j} \lambda_i \mu_j a_i \odot b_j \quad (2.2) \\
\lambda f &= \sum_{i} \lambda \lambda_i a_i \quad (2.3) \\
f^* &= \sum_{i} \bar{\lambda}_i a_i \quad (2.4)
\end{align*}
\]

It is easy to see that these operations are invariant under the relation \( \sim \) i.e. \( f \sim g \) implies \( f + h \sim g + h, fh \sim gh \) and \( \lambda f \sim \lambda g \). For example, if \( f, g \) and \( h \) are as above and \( f \sim g \), then

\[
f + h = \sum a_i (\lambda_i + \nu_k) a_i \odot c_k \quad \text{and} \quad g + h = \sum b_j (\mu_j + \nu_k) b_j \odot c_k\]

If \( (a_i \odot c_k) \odot (b_j \odot c_k') \neq 0 \), then \( a_i \odot b_j \neq 0 \) and \( k = k' \). This implies that \( \lambda_i = \mu_j \), and from this we conclude that \( f + h \sim g + h \).

Define the semi-norm \( \| \| \) on \( B' \) by setting \( \| f \| = \max \{ | \lambda(a) | : a \in [f], a \neq 0 \} \). Also it is easy to see that \( \| \| \) is invariant under the relation \( \sim \) i.e. \( f \sim g \) implies \( \| f \| = \| g \| \).

Let \( \tilde{B} \) be the collection of all equivalence classes with respect to \( \sim \). Then \( \tilde{B} \) is a normed linear algebra with respect to the operations induced by operations on \( B' \) (we shall use same notation), the additive identity \( 0 \) of \( \tilde{B} \) is the set of all members of \( B' \) of the form \( f = \sum \lambda_i a_i \),
where \( \{a_1, \ldots, a_n\} \) is some decomposition of identity and \( \lambda_i = 0 \) for \( i = 1, \ldots, n \). Also \( \| f \| = 0 \) if and only if \( f = \overline{0} \), and it is not difficult to show that \( \| fg \| \leq \| f \| \cdot \| g \| \) for all \( f, g \in \mathcal{B} \).

Let \( \mathcal{B} \) be the completion of \( \tilde{\mathcal{B}} \) with respect to \( \| \| \). Then \( \mathcal{B} \) is a commutative \( B^* \)-algebra with identity.

Apply Representation Theorem of Gelfand to \( \mathcal{B} \): There exists a compact Hausdorff space \( S \) such that \( \mathcal{B} \) is \( * \)-isomorphic and isometric to the algebra \( C(S) \) of all continuous complex valued functions on \( S \). For each \( f \in \mathcal{B} \) let \( \hat{f}(s) \) denote the corresponding image of \( f \) under this isomorphism. Isometry between \( \mathcal{B} \) and \( C(S) \) means that \( \| f \| = \sup_{s \in S} |\hat{f}(s)| \).

Now note that there is a natural imbedding of the Boolean ring \( \mathcal{A} \) into \( \mathcal{B} \): for each \( a \in \mathcal{A} \) let \( f_a = 1 \cdot a + 0 \cdot a' \), where \( a' = 1 \oplus a \) is the complement of \( a \) in \( \mathcal{A} \). (Note that \( \mathcal{A} \) has also a structure of a Boolean algebra (see Appendix Three of Simmons [7]).) Then \( f_a f_a = f_a \), from which we conclude that \( \hat{f}_a(s) \) assumes either 1 or 0 at any \( s \in S \). Let \( A = \{ s \in S : f_a(s) = 1 \} \), then \( \hat{f}_a(s) \) is the characteristic function of \( A \) i.e. \( \hat{f}_a = \varphi_A \), and it follows from continuity of \( \hat{f}_a \) that \( A \) is both open and closed in \( S \). The correspondence \( a \leftrightarrow A \) is 1-1 and preserves both lattice and algebraic operations of \( \mathcal{B} \). A simplest way to establish this is to show that this correspondence preserves multiplication and complementation. But both facts follow easily from the identities \( "f_a \oplus b = f_a f_b" \) and \( "f_a f_a' = 0" \) (\( a, b \in \mathcal{A} \) and \( a' = a \oplus 1 \)).

It remains to show that \( S \) is totally disconnected. Since \( S \) is a Hausdorff space, we need only to show that for any open neighborhood \( U \) of \( s_0 \in S \) there exists an open-closed set \( O \) such that \( s_0 \in O \subset U \). Since every compact \( T_2 \) space is normal, there exists a continuous real valued function \( x(s) \) (a member of \( C(S) \)) such that \( x(s_0) = 0 \), \( x(S \sim U) = 1 \) and \( 0 \leq x(s) \leq 1 \) everywhere else (see Urysohn's Lemma in Sec. 3, Chap. 8 of Royden [6] or 3C in Loomis [3]). Let \( x \in \mathcal{B} \) be such that \( \hat{x}(s) = x(s) \) and let \( f \in \mathcal{B}' \), \( f = \sum_{i=1}^{n} \lambda_i a_i \), be such that \( \| f - x \| < \frac{1}{4} \). From "\( \sum_{i=1}^{n} \overline{a}_i = 1 \) and \( a_i \oplus a_j = 0 \) if \( i \neq j \)" we conclude that sets \( A_1, A_2, \ldots, A_n \) (corresponding to \( a_1, a_2, \ldots, a_n \) under above discussed correspondence \( a \leftrightarrow A \)) are disjoint and \( \cup A_i = S \). Hence there exists exactly one index \( j \in \{1, 2, \ldots, n\} \) such that \( s_0 \in A_j \). The set \( A_j \) is both open and closed and \( s \in A_j \) implies \( \hat{f}(s) = \lambda_j = \hat{f}(s_0) \), from which we conclude that \( A_j \subset U \): if \( s \in A_j \), then \( |x(s)| \leq |x(s) - \hat{f}(s)| + |\hat{f}(s_0) - x(s_0)| + |x(s_0)| < 1 \), and this implies that \( s \in U \).

To see that each open-closed subset \( A \) of \( S \) corresponds to some \( a \in \mathcal{A} \) we use compactness of \( S \), which implies compactness of \( A \). As above, for each \( s \in A \) we select an open-closed set \( O_s \) such that \( s \in O_s \subset A \). Compactness of \( A \) implies that there is a finite set \( \{A_1, \ldots, A_m\} \) of open-closed subsets of \( S \), each corresponding to some \( a_i \in \mathcal{A} \) \( (i = 1 \ldots m) \), such that \( A = \bigcup_{i=1}^{m} A_i \). This implies that \( A \) corresponds to some \( a \in \mathcal{A} \) \( (a = a_1 \cup a_2 \ldots \cup a_m) \).

REFERENCES


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