Internat. J. Math. & Math. Sci.
VOL. 18 NO. 3 (1995) 469-474

SOME PROPERTIES OF STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC-CONJUGATE POINTS

HASSOON AL-AMIRI

Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, Ohio 43403

DAN COMAN

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109

PETRU T. MOCANU

Faculty of Mathematics
Babes-Bolyai University
3400 Cluj-Napoca, Romania

(Received August 18, 1993 and in revised form February 3, 1994)

ABSTRACT. Let A be the class of all analytic functions in the unit disk U such that $f(0) = f'(0) - 1 = 0$. A function $f \in A$ is called starlike with respect to $2n$ symmetric-conjugate points if $Re\ zf'(z)/f_n(z) > 0$ for $z \in U$, where

$$f_n(z) = \frac{1}{2n}\sum_{k=0}^{n-1}[\omega^{-k}f(\omega^k z) + \omega^k \overline{f(\omega^k \bar{z})}],$$

$\omega = exp(2\pi i/n]$. This class is denoted by S_n^* and was studied in [1]. A sufficient condition for starlikeness with respect to symmetric-conjugate points is obtained. In addition, images of some subclasses of S_n^* under the integral operator $I : A \to A, I(f) = F$ where

$$F(z) = \frac{c+1}{(g(z))^c}\int_0^z f(t)(g(t))^{c-1}g'(t)dt, \quad c > 0$$

and $g \in A$ is given are determined.

KEY WORDS AND PHRASES: symmetric-conjugate points; starlike; differential subordinations; integral operator; strongly starlike; α-convex.

1991 MATHEMATICS SUBJECT CLASSIFICATION: Primary 30C45.

1. INTRODUCTION

Let $m \geq 1$ be an integer and let A_m be the class of all functions f that are analytic in the unit disk U and having the power series expansion of the form

$$f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \ldots, \quad z \in U.$$

We set $A \equiv A_1$.

In [1] the concept of starlike functions with respect to $2n$ symmetric-conjugate points was introduced. We recall that for a positive integer n and for $\omega = exp(2\pi i/n)$, a function $f \in A$ is called a starlike function with respect to $2n$ symmetric-conjugate points if

$$Re z f'(z)/f_n(z) > 0, \quad z \in U,$$

where

$$f_n(z) = \frac{1}{2n}\sum_{k=0}^{n-1}[\omega^{-k} f(\omega^k z) + \omega^k \overline{f(\omega^k \bar{z})}]. \tag{1.1}$$

The class of all such function is denoted by S_n^*. Note that $S_n^* \subseteq C$, where C is the class of close-to-convex functions.

The following relations can be deduced from (1.1).

$$f_n' = \frac{1}{2n}\sum_{k=0}^{n-1}[f'(\omega^k z) + \overline{f'(\omega^k \bar{z})}], \tag{1.2}$$

$$f''(z) = \frac{1}{2n}\sum_{k=0}^{n-1}[\omega^k f''(\omega^k z) + \omega^{-k}\overline{f''(\omega^k \bar{z})}], \tag{1.3}$$

$$\begin{aligned} f_n(\omega^j z) &= \omega^j f_n(z), \quad f_n(\bar{z}) = \overline{f_n(\bar{z})}, \\ f_n'(\omega^j z) &= f_n'(z), \quad f_n'(\bar{z}) = \overline{f_n'(z)}. \end{aligned} \tag{1.4}$$

In this paper we shall determine a sufficient condition for starlikeness with respect to symmetric-conjugate points. In addition, we find the images of certain subclasses of S_n^* under the integral operator $I : A \to A, I(f) = F$ where,

$$F(z) = \frac{c+1}{(g(z))^c}\int_0^z f(t)(g(t))^{c-1} g'(t)dt, \tag{1.5}$$

$c \geq 0$ and $g \in A$ is a given function. The case $g(z) \equiv z$ was discussed in [1]. A more general integral operator was studied in [2].

2. PRELIMINARIES

In order to prove our main results, we need the following definitions and lemmas. Let us first recall the definition of subordination. If $f, g \in A$ and g is univalent, f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subset g(U)$. Also, a function $f \in A$ is called strongly starlike of order $\alpha, \alpha \in (0, 1]$ if

$$zf'(z)/f(z) \prec ((1+z)/(1-z))^\alpha$$

The class of all such functions is denoted by $S^*(\alpha)$. A function $f \in A$ is called α -convex, $\alpha \in R$ if

$$Re[(1-\alpha)zf'(z)/f(z) + \alpha((zf''(z)/f'(z)) + 1)] > 0,$$

$z \in U$. The class of all such functions is denoted by M_α.

LEMMA 2.1. *[4] Let $m \geq 1$ be an integer and*

$$p(z) = 1 + p_m z^m + p_{m+1} z^{m+1} + \ldots, \quad z \in U, \tag{2.1}$$

be analytic in U. If the function p is not with positive real part in U, then there is a point $z_0 \in U$ such that $p(z_0) = is$, $z_0 p'(z_0) = t$, where s, t are real and $t \leq -m(1+s^2)/2$.

LEMMA 2.2. *If $f \in A_m$ satisfies*

$$|f''(z)/f'(z)| \leq 1 + m/2, \quad z \in U,$$

then for all $z \in U$

i) $Re\ f(z)/(zf'(z)) > 1/2$,

ii) $|(zf'(z)/f(z)) - 1| < 1$

PROOF. It is clear that (i) and (ii) are equivalent. Let $p(z) = 2f(z)/(zf'(z)) - 1$. Then p has the form (2.1) and

$$zf''(z)/f'(z) = (1 - p(z) - zp'(z))/(p(z) + 1).$$

Suppose p is not with a positive real part. Then by Lemma 2.1 there is a $z_0 \in U$ such that $p(z_0) = is$, $z_0 p'(z_0) = t$, where $t \leq -m(1+s^2)/2$. Consequently,

$$\begin{aligned} |z_0 f''(z_0)/f'(z_0)|^2 &= ((1-t)^2 + s^2)/(1+s^2) \\ &\geq [(1 + m(1+s^2)/2)^2 + s^2]/(1+s^2) \\ &\geq (1 + m/2)^2, \end{aligned}$$

which contradicts the hypothesis of this lemma. The proof is now complete. The case $m = 1$ of Lemma 2.2 can be found in [5].

LEMMA 2.3. *[2] Let $\alpha \in (0, 1]$. For $c = 0$ suppose that $g \in S^\star(1-\alpha)$, while $g \in M_{1/c}$, for $c > 0$. If the function $f \in A$ satisfies*

$$g(z)f'(z)/(g'(z)f(z)) \prec ((1+z)/(1-z))^\alpha$$

then the function F defined by (1.5) is also in A, $F(z)/z \neq 0$ for $z \in U$ and

$$g(z)F'(z)/(g'(z)F(z)) \prec ((1+z)/(1-z))^\alpha.$$

LEMMA 2.4. *[3] Let $P(z)$ be analytic function in U with $Re\ P(z) > 0$, $z \in U$, and let h be a convex function in U. If p is analytic in U with $p(0) = h(0)$, then*

$$p(z) + P(z)zp'(z) \prec h(z) \quad \textit{implies} \quad p(z) \prec h(z).$$

3. MAIN RESULTS.

THEOREM 3.1. *Let $f \in A_m$, $m \geq 2$, and let n be a positive integer. If*

$$|f''(z)/f_n'(z)| \leq (m^2 - 1)/(4m), \tag{3.1}$$

$z \in U$, where $f_n(z)$ is defined by (1.2), then $f \in S_n^\star$

PROOF. From (1.4) and (3.1) we deduce

$$|\omega^k f''(\omega^k z)/f_n'(z)| \leq (m^2-1)/(4m),$$

and

$$|\omega^{-k}\overline{f''(\omega^k \bar{z})}/f_n'(z)| \leq (m^2-1)/(4m).$$

Combining these relations with (1 3) to get

$$|f_n''(z)/f_n'(z)| \leq (m^2-1)/(4m), \quad z \in U.$$

Since $(m^2-1)/(4m) \leq 1+m/2$, then Lemma 2.2 can be applied to f_n to deduce, in particular, $f_n(z)/z \neq 0$ for $z \in U$. To complete the proof, let $p(z) = zf'(z)/f_n(z)$, then we need to show that $Re\ p(z) > 0$. Note that since f and f_n are in A_m, so p has the form (2.1) for $m \geq 1$. In addition

$$zf''(z)/f_n'(z) = (f_n(z)/(zf_n'(z)))(zp'(z) + p(z)(zf_n'(z)/f_n(z) - 1)).$$

Assume p is not with positive real part in U. Then by Lemma 1.1, there is a point $z_0 \in U$ such that $p(z_0) = is$, $z_0p'(z_0) = t$ and $t \leq -m(1+s^2)/2$. Using the conclusions of Lemma 2.2 for f_n, we obtain

$$\begin{aligned}
|z_0f''(z_0)/f_n'(z_0)| &\geq 1/2|t + is(z_0f_n'(z_0)/f_n(z_0) - 1)| \\
&\geq 1/2(|t| - |s|) \\
&\geq 1/2(m(1+s^2)/2 - |s|) \\
&\geq (m^2-1)/(4m),
\end{aligned}$$

which contradicts the hypothesis (3.1). Hence $f \in S_n^\star$. This completes the proof of this theorem.

THEOREM 3.2. *Suppose $\alpha \in (0,1]$, $c \geq 0$ and $n \geq 1$ is an integer. Let $g \in S^\star(1-\alpha)$ be a function with the power series expansion*

$$g(z) = z + g_1z^{n+1} + g_2z^{2n+1} + \ldots,$$

$z \in U$, where all the coefficients g_j are real. In addition, suppose that $g \in M_{1/c}$ for $c > 0$. Consider the integral operator $I : A \to A$, $I(f) = F$, where F is given by (1.5). If

$$g(z)f'(z)/(g'(z)f_n(z)) \prec ((1+z)/(1-z))^\alpha, \tag{3.2}$$

then

$$g(z)F'(z)/(g'(z)F_n(z)) \prec ((1+z)/(1-z))^\alpha,$$

where f_n and F_n are the functions associated with f and F as given in (1.1), respectively.

PROOF. First, we show that $F_n = I(f_n)$. Using (1.5) one can easily write $F(z)$ in the following form:

$$F(z) = \frac{c+1}{(g(z)/z)^c} \int_0^1 f(xz)(g(xz)/(xz))^{c-1} g'(xz) x^{c-1} dx.$$

From the expansion form of $g(z)$, it follows that

$$\frac{1}{2n}\omega^{-k} F(\omega^k z) = \frac{c+1}{(g(z)/z)^c} \int_0^1 \frac{1}{2n}\omega^{-k} f(\omega^k xz)(g(xz)/xz)^{c-1} g'(xz) x^{c-1} dx,$$

and

$$\frac{1}{2n}\omega^k \overline{F(\omega^k \bar{z})} = \frac{c+1}{(g(z)/z)^c} \int_0^1 \frac{1}{2n}\omega^k \overline{f(\omega^k x\bar{z})}(g(xz)/(xz))^{c-1} g'(xz)^{c-1} x dx.$$

Now by summation and (1.1) we deduce easily that $F_n = I(f_n)$. Replacing z by $\omega^k z$ and then by $\omega^k \bar{z}$, $k = \{0, 1, ..., n-1\}$ in (3.2) and using the relations (1.2) and (1.4) and also the fact that

$$g(\omega^k z) = \omega^k g(z), \quad g(\omega^k \bar{z}) = \omega^k \overline{g(z)}, \quad g'(\omega^k z) = g'(z), \quad g'(\omega^k \bar{z}) = \overline{g'(z)}.$$

We deduce the relation

$$g(z) f_n'(z)/(g'(z) f_n(z)) \prec ((1+z)/(1-z))^\alpha.$$

Applying Lemma 2.3 to the above to get

$$arg(G(z) z F_n'(z)/F_n(z) + c) < \alpha\pi/2, \tag{3.3}$$

where

$$G(z) = g(z)/(zg'(z)).$$

Let

$$P(z) = G(z)(G(z) z F_n'(z)/F_n(z) + c)^{-1}. \tag{3.4}$$

From (3.3) and the fact that $g \in S^\star(1-\alpha)$, we easily deduce from (3.4) that

$$Re\ P(z) > 0.$$

Let

$$p(z) = g(z) F'(z)/(g'(z) F_n(z)).$$

Lemma 2.3 shows that $p(z)$ is analytic in U. Hence multiplication of (1.5) by g^c and differentiating the new equation we obtain

$$G(z) z F'(z) + cF(z) = (c+1) f(z) \tag{3.5}$$

and

$$G(z)zF_n'(z) + cF_n(z) = (c+1)f_n(z). \tag{3.6}$$

Substituting in (3 5)

$$G(z)F'(z) = p(z)F_n(z)$$

then differentiating the new equation and using (3.6) to get

$$\begin{aligned} p(z) + P(z)zp'(z) &= g(z)f'(z)/(g'(z)f_n(z)) \\ &\prec ((1+z)/(1-z))^{\alpha}, \end{aligned} \tag{3.7}$$

where $P(z)$ is given by (3.4) with $Re\ P(z) > 0$. Applying Lemma 2.4 to (3.7) to deduce

$$Re\ p(z) = Re\ g(z)F'(z)/(g'(z)F_n(z)) > 0.$$

This completes the proof of this theorem.

ACKNOWLEDGEMENT. This work was completed while the third author was a visiting professor at Bowling Green State University in the Fall of 1992.

REFERENCES

1. Al-Amiri, H. S., Coman, D., and Mocanu, P. T., On certain classes of starlike functions with respect to symmetric-conjugate points, submitted.

2. Coman, D., On a generalized type of Briot-Bouquet differential subordination, *Mathematica*, (to appear).

3. Miller, S. S. and Mocanu, P. T., Generalized second order inequalities in the complex plane, "Babes-Bolyai" University Fac. of Math., Research Seminars, *Seminar on Geometric Function Theory Preprint* **4**(1982), 96-114.

4. Miller, S. S. and Mocanu, P. T., The theory and applications of second order differential subordinations, *Studia Univ. "Babes-Bolyai", Math.*, **4**(1989), 3-33.

5. Mocanu, P. T., Some integral operators and starlike functions, *Rev, Roumain Math. Pures Appl* **31**(1986), 231-235.