SOME PROPERTIES OF STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC-CONJUGATE POINTS

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ABSTRACT. Let $A$ be the class of all analytic functions in the unit disk $U$ such that $f(0) = f'(0) - 1 = 0$. A function $f \in A$ is called starlike with respect to $2n$ symmetric-conjugate points if $\Re zf'(z)/f_n(z) > 0$ for $z \in U$, where

$$f_n(z) = \frac{1}{2n} \sum_{k=0}^{n-1} [\omega^{-k} f(\omega^k z) + \omega^k f(\overline{\omega^k z})],$$

$\omega = \exp(2\pi i/n)$. This class is denoted by $S^*_n$ and was studied in [1]. A sufficient condition for starlikeness with respect to symmetric-conjugate points is obtained. In addition, images of some subclasses of $S^*_n$ under the integral operator $I : A \rightarrow A$, $I(f) = F$ where

$$F(z) = \frac{c+1}{(g(z))^c} \int_0^z f(t)(g(t))^{c-1} g'(t) dt, \quad c > 0$$

and $g \in A$ is given are determined.

KEY WORDS AND PHRASES: symmetric-conjugate points; starlike; differential subordinations; integral operator; strongly starlike; $\alpha$-convex.

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1. INTRODUCTION

Let $m \geq 1$ be an integer and let $A_m$ be the class of all functions $f$ that are analytic in the unit disk $U$ and having the power series expansion of the form

$$f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \ldots, \quad z \in U.$$
We set \( A_1 = A_1 \).

In [1] the concept of starlike functions with respect to \( 2n \) symmetric-conjugate points was introduced. We recall that for a positive integer \( n \) and for \( \omega = \exp(2\pi i/n) \), a function \( f \in A \) is called a starlike function with respect to \( 2n \) symmetric-conjugate points if

\[
\text{Re} \left( \frac{zf'(z)}{f_n(z)} \right) > 0, \quad z \in U,
\]

where

\[
f_n(z) = \frac{1}{2n} \sum_{k=0}^{n-1} [\omega^{-k} f(\omega^k z) + \omega^k \overline{f(\omega^k \bar{z})}]. \tag{1.1}
\]

The class of all such functions is denoted by \( S_n^* \). Note that \( S_n^* \subseteq C \), where \( C \) is the class of close-to-convex functions.

The following relations can be deduced from (1.1).

\[
f'_n = \frac{1}{2n} \sum_{k=0}^{n-1} [f'(\omega^k z) + \overline{f'(\omega^k \bar{z})}], \tag{1.2}
\]

\[
f''(z) = \frac{1}{2n} \sum_{k=0}^{n-1} [\omega^k f''(\omega^k z) + \omega^{-k} \overline{f''(\omega^k \bar{z})}], \tag{1.3}
\]

\[
f_n(\omega^j z) = \omega^j f_n(z), \quad f_n(\bar{z}) = \overline{f_n(z)},
\]

\[
f'_n(\omega^j z) = f'_n(z), \quad f'_n(\bar{z}) = \overline{f'_n(z)}. \tag{1.4}
\]

In this paper we shall determine a sufficient condition for starlikeness with respect to symmetric-conjugate points. In addition, we find the images of certain subclasses of \( S_n^* \) under the integral operator \( I : A \to A, I(f) = F \) where,

\[
F(z) = \frac{c + 1}{(g(z))^c} \int_0^z f(t)(g(t))^{c-1} g'(t)dt, \tag{1.5}
\]

\( c \geq 0 \) and \( g \in A \) is a given function. The case \( g(z) = z \) was discussed in [1]. A more general integral operator was studied in [2].

2. PRELIMINARIES

In order to prove our main results, we need the following definitions and lemmas. Let us first recall the definition of subordination. If \( f, g \in A \) and \( g \) is univalent, \( f \) is subordinate to \( g \), written \( f \prec g \), or \( f(z) \prec g(z) \), if \( f(0) = g(0) \) and \( f(U) \subset g(U) \). Also, a function \( f \in A \) is called strongly starlike of order \( \alpha, \alpha \in (0, 1] \) if

\[
zf'(z)/f(z) \prec ((1 + z)/(1 - z))^\alpha
\]

The class of all such functions is denoted by \( S^*(\alpha) \). A function \( f \in A \) is called \( \alpha \)-convex, \( \alpha \in R \) if

\[
\text{Re}[(1 - \alpha)zf'(z)/f(z) + \alpha((zf''(z)/f'(z)) + 1)] > 0,
\]

\( z \in U \). The class of all such functions is denoted by \( M_\alpha \).
LEMMA 2.1. \[ [4] \] Let \( m \geq 1 \) be an integer and
\[
p(z) = 1 + p_m z^m + p_{m+1} z^{m+1} + \ldots, \quad z \in U,
\] be analytic in \( U \). If the function \( p \) is not with positive real part in \( U \), then there is a point \( z_0 \in U \) such that \( p(z_0) = is, \quad z_0 p'(z_0) = t \), where \( s, t \) are real and \( t \leq -m(1 + s^2)/2 \).

LEMMA 2.2. If \( f \in A_m \) satisfies
\[
|f''(z)/f'(z)| \leq 1 + m/2, \quad z \in U,
\] then for all \( z \in U \)

i) \( \text{Re } f(z)/(zf'(z)) > 1/2 \),

ii) \( |(zf'(z)/f(z)) - 1| < 1 \)

PROOF. It is clear that (i) and (ii) are equivalent. Let \( p(z) = 2f(z)/(zf'(z)) - 1 \). Then \( p \) has the form (2.1) and
\[
z f''(z)/f'(z) = (1 - p(z) - zp'(z))/(p(z) + 1).
\]
Suppose \( p \) is not with a positive real part. Then by Lemma 2.1 there is a \( z_0 \in U \) such that \( p(z_0) = is, \quad z_0 p'(z_0) = t \), where \( t \leq -m(1 + s^2)/2 \). Consequently,
\[
|z_0 f''(z)/f'(z)|^2 = ((1 - t)^2 + s^2)/(1 + s^2) \\
\geq [(1 + m(1 + s^2)/2)^2 + s^2]/(1 + s^2) \\
\geq (1 + m/2)^2,
\]
which contradicts the hypothesis of this lemma. The proof is now complete. The case \( m = 1 \) of Lemma 2.2 can be found in [5].

LEMMA 2.3. \[ [2] \] Let \( \alpha \in (0, 1] \). For \( c \in (0, 1/2) \), suppose that \( g \in S^*(1 - \alpha) \), while \( g \in M_1/c \), for \( c > 0 \). If the function \( f \in A \) satisfies
\[
g(z)f'(z)/(g'(z)f(z)) < ((1 + z)/(1 - z))^\alpha
\]
then the function \( F \) defined by (1.5) is also in \( A \), \( F(z)/z \neq 0 \) for \( z \in U \) and
\[
g(z)F''(z)/(g'(z)F(z)) < ((1 + z)/(1 - z))^\alpha.
\]

LEMMA 2.4. \[ [3] \] Let \( P(z) \) be analytic function in \( U \) with \( \text{Re } P(z) > 0, \quad z \in U \), and let \( h \) be a convex function in \( U \). If \( p \) is analytic in \( U \) with \( p(0) = h(0) \), then
\[
p(z) + P(z)zp'(z) < h(z) \quad \text{implies} \quad p(z) < h(z).
\]

3. MAIN RESULTS.

THEOREM 3.1. Let \( f \in A_m, \quad m \geq 2 \), and let \( n \) be a positive integer. If
\[
|f''(z)/f_1(z)| \leq (m^2 - 1)/(4m), \quad (3.1)
\]
\( z \in U \), where \( f_n(z) \) is defined by (1.2), then \( f \in S^*_n \)

PROOF. From (1.4) and (3.1) we deduce
\[ |\omega^k f''(\omega^k z)/f_n'(z)| \leq (m^2 - 1)/(4m), \]

and

\[ |\omega^{-k} f''(\omega^k z)/f_n'(z)| \leq (m^2 - 1)/(4m). \]

Combining these relations with (1.3) to get

\[ |f_n''(z)/f_n'(z)| \leq (m^2 - 1)/(4m), \quad z \in U. \]

Since \( (m^2 - 1)/(4m) \leq 1 + m/2 \), then Lemma 2.2 can be applied to \( f_n \) to deduce, in particular, \( f_n(z)/z \neq 0 \) for \( z \in U \). To complete the proof, let \( p(z) = z f''(z)/f_n(z) \), then we need to show that \( \text{Re } p(z) > 0 \). Note that since \( f \) and \( f_n \) are in \( A_m \), so \( p \) has the form (2.1) for \( m \geq 1 \). In addition

\[ z f''(z)/f_n'(z) = (f_n(z)/(z f_n'(z)))(zp'(z) + p(z)(z f_n'(z)/f_n(z) - 1)). \]

Assume \( p \) is not with positive real part in \( U \). Then by Lemma 1.1, there is a point \( z_0 \in U \) such that \( p(z_0) = is, z_0 p'(z_0) = t \) and \( t \leq -m(1 + s^2)/2 \). Using the conclusions of Lemma 2.2 for \( f_n \), we obtain

\[ |z_0 f''(z_0)/f_n'(z_0)| \geq 1/2|t + is(z_0 f_n'(z_0)/f_n(z_0) - 1)| \]
\[ \geq 1/2(|t| - |s|) \]
\[ \geq 1/2(m(1 + s^2)/2 - |s|) \]
\[ \geq (m^2 - 1)/(4m), \]

which contradicts the hypothesis (3.1). Hence \( f \in S_n^* \). This completes the proof of this theorem.

**THEOREM 3.2.** Suppose \( \alpha \in (0, 1], c \geq 0 \) and \( n \geq 1 \) is an integer. Let \( g \in S^*(1 - \alpha) \) be a function with the power series expansion

\[ g(z) = z + g_2 z^{n+1} + g_2 z^{2n+1} + \ldots, \]

where all the coefficients \( g_j \) are real. In addition, suppose that \( g \in M_{1/c} \) for \( c > 0 \). Consider the integral operator \( I : A \rightarrow A, I(f) = F \), where \( F \) is given by (1.5). If

\[ g(z)f'(z)/(g'(z)f_n(z)) \prec ((1 + z)/(1 - z))^\alpha, \]

then

\[ g(z)F'(z)/(g'(z)F_n(z)) \prec ((1 + z)/(1 - z))^\alpha, \]

where \( f_n \) and \( F_n \) are the functions associated with \( f \) and \( F \) as given in (1.1), respectively.

**PROOF.** First, we show that \( F_n = I(f_n) \). Using (1.5) one can easily write \( F(z) \) in the following form:
\[ F(z) = \frac{c+1}{(g(z)/z)^c} \int_0^1 f(xz)(g(xz)/(xz))^{c-1} g'(xz) x^{c-1} dx. \]

From the expansion form of \( g(z) \), it follows that
\[
\frac{1}{2n} \omega^{-k} F'(\omega^k z) = \frac{c+1}{(g(z)/z)^c} \int_0^1 \frac{1}{2n} \omega^{-k} f(\omega^k x z)(g(xz)/(xz))^{c-1} g'(xz) x^{c-1} dx,
\]
and
\[
\frac{1}{2n} \omega^{k} F'(\omega^k z) = \frac{c+1}{(g(z)/z)^c} \int_0^1 \frac{1}{2n} \omega^{k} f(\omega^k x z)(g(xz)/(xz))^{c-1} g'(xz)x^{c-1} dx.
\]

Now by summation and (1.1) we deduce easily that \( F_n = I(f_n) \). Replacing \( z \) by \( \omega^k z \) and then by \( \omega^k z, k = \{0, 1, ..., n-1\} \) in (3.2) and using the relations (1.2) and (1.4) and also the fact that
\[ g(\omega^k z) = \omega^k g(z), \quad g'(\omega^k z) = g'(z), \quad g''(\omega^k z) = \overline{g'(z)}. \]

We deduce the relation
\[ g(z)f'_n(z)/(g'(z)f_n(z)) < ((1+z)/(1-z))^\alpha. \]

Applying Lemma 2.3 to the above to get
\[ \arg(G(z)zF'_n(z)/F_n(z) + c) < \alpha \pi/2, \quad (3.3) \]
where
\[ G(z) = g(z)/(zg'(z)). \]

Let
\[ P(z) = G(z)(G(z)zF'_n(z)/F_n(z) + c)^{-1}. \quad (3.4) \]

From (3.3) and the fact that \( g \in S^*(1-\alpha) \), we easily deduce from (3.4) that
\[ Re \, P(z) > 0. \]

Let
\[ p(z) = g(z)F'(z)/(g'(z)F_n(z)). \]

Lemma 2.3 shows that \( p(z) \) is analytic in \( U \). Hence multiplication of (1.5) by \( g^c \) and differentiating the new equation we obtain
\[ G(z)zF'(z) + cF(z) = (c+1)f(z) \quad (3.5) \]
474 H. AL-AMIRI, D. COMAN AND P. T. MOCANU

and

\[ G(z)z F''_n(z) + c F'_n(z) = (c + 1) f_n(z). \]  

(3.6)

Substituting in (3.5)

\[ G(z)F'(z) = p(z) F'_n(z) \]

then differentiating the new equation and using (3.6) to get

\[ p(z) + P(z) z p'(z) = g(z) f'(z)/(g'(z) f_n(z)) \]

\[ -< ((1 + z)/(1 - z))^\alpha, \]  

(3.7)

where \( P(z) \) is given by (3.4) with \( \text{Re} P(z) > 0 \). Applying Lemma 2.4 to (3.7) to deduce

\[ \text{Re} p(z) = \text{Re} g(z) F'(z)/(g'(z) F_n(z)) > 0. \]

This completes the proof of this theorem.

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