GLIDING HUMP PROPERTIES AND SOME APPLICATIONS

JOHANN BOOS DANIEL J. FLEMING

JOHANN BOOS

Fachbereich Mathematik
Emil-Fischer-Platz 1
30880 Hagen
Germany

DANIEL J. FLEMING

Department of Mathematics
St. Lawrence University
Canton, NY 13607
USA

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ABSTRACT. In this note we consider several types of gliding hump properties for a sequence space $E$ and we consider the various implications between these properties. By means of examples we show that most of the implications are strict and they afford a sort of structure between solid sequence spaces and those with weakly sequentially complete $\beta$-duals. Our main result is used to extend a result of Bennett and Kalton which characterizes the class of sequence spaces $E$ with the property that $E \subset S_F$ whenever $F$ is a separable FK space containing $E$ where $S_F$ denotes the sequences in $F$ having sectional convergence. This, in turn, is used to identify a gliding humps property as a sufficient condition for $E$ to be in this class.

KEY WORDS AND PHRASES. Gliding hump properties, weak sequential completeness of the $\beta$-dual, sectional convergence in FK spaces, Theorem of Schur, Theorem of Hahn.

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1. INTRODUCTION.

Over the past eighty years the “gliding hump” technique has been a frequently used tool to establish results in summability and sequence space theory. Among the more familiar examples would be the Silverman-Toeplitz theorem which gives necessary and sufficient conditions for the regularity of a summability method [22], the Mazur-Orlicz bounded consistency theorem ([6], [12] and [13]), the theorem of Köthe and Toeplitz on the weak sequential completeness of the Köthe dual of a solid sequence space [11] and the theorems of Schur on the characterization of coercive matrices and the equivalence of weak and strong convergence in $\ell_1$ [19]. Whereas the first three of these have subsequently been argued using functional analytic techniques (see e.g. [24] and [10]) no such “soft” proofs of Schur’s theorems are known. Various authors have considered sequence spaces enjoying certain gliding hump type properties. See for example, [8] for extensions of Schur’s theorems, ([4], [5], [20]) for Mazur-Orlicz type theorems and ([5], [14]) for weak sequential completeness results. The gliding hump technique has also proven to be a key ingredient in the solution to problems related to the Wilansky Property ([1], [21], [15]).

In section 3 of this note we introduce various types of gliding hump properties and discuss the implications between them. We give examples in section 5 to show that most of these implications are strict and they are, in some sense, affording a structure to the set of sequence spaces between the solid spaces and those with weakly sequentially complete $\beta$-duals.
In [2, Theorem 6], Bennett and Kalton characterized the class of sequence spaces $E$ for which $E \subseteq S_F$ whenever $F$ is a separable FK space containing $E$ (here, $S_F$ denotes the elements of $F$ having sectional convergence). In Theorem 3.6 we extend their result by showing that it suffices to consider only the case where $F$ is a convergence domain of a matrix. Combining this observation with our main result Theorem 3.5 we obtain in Corollary 3.7 the more tractable pointwise weak gliding hump property (see definition 3.1 below) as a sufficient condition for $E$ to belong to this class. In section 4 we apply the techniques of this paper to obtain short proofs of some classical results.

2. NOTATION AND PRELIMINARIES.

Let $\omega$ denote the linear space of all scalar (real or complex) sequences. By a sequence space $E$ we shall mean any linear subspace of $\omega$. A sequence space $E$ endowed with a locally convex topology is called a $K$-space if the inclusion map $i : E \rightarrow \omega$ is continuous where $\omega$ has the topology of coordinatewise convergence. A $K$-space $E$ with a Fréchet topology is called an FK-space. If, in addition, the topology is normable then $E$ is called a BK-space. We assume throughout this note familiarity with the standard sequence spaces and their natural topologies (see e.g. [24], [9]).

For a sequence space $E$ the multiplier space of $E$ and the $\beta$-dual of $E$ are given by

$$
\mathcal{M}(E) = \left\{ x \in \omega \mid xy \in E \text{ for each } y \in E \right\}
$$

and

$$
E^\beta = \left\{ x \in \omega \mid \sum_k x_k y_k \text{ converges for each } y \in E \right\}
$$

where $xy$ denotes the coordinatewise product. For $x \in \omega$, $n \in \mathbb{N}$ the $n$th section of $x$ is

$$
x^{[n]} = \sum_{k=1}^{n} x_k e^k
$$

where $e^k = (\delta_{nk})_{n=1}^\infty$ is the $k$th coordinate vector. For any positive term sequence $\mu = (\mu_k)$ let

$$
E_\mu = \left\{ x \in \omega \mid \left( \frac{x_k}{\mu_k} \right) \in E \right\}.
$$

If $(E, F)$ is a dual pair then $\sigma(E, F)$, $\tau(E, F)$ denotes the weak topology and the Mackey topology respectively. For a sequence space $E$ and a linear subspace $F$ of $E^\beta$, $(E, F)$ is a dual pair under the natural bilinear form

$$
\langle x, y \rangle = \sum_k x_k y_k.
$$

If $E$ is a $K$-space containing $\varphi$, the space of finitely non-zero sequences, we let

$$
L_E = \left\{ x \in E \mid \{x^{[n]}\}_{n \in \mathbb{N}} \text{ is bounded in } E \right\}
$$

$$
W_E = \left\{ x \in E \mid x^{[n]} \rightarrow x \sigma(E, E') \right\}
$$

$$
S_E = \left\{ x \in E \mid x^{[n]} \rightarrow x \text{ in } E \right\}
$$

where $E'$ denotes the topological dual of $E$. A $K$-space $E$ containing $\varphi$ with $E = S_E$ is called an AK-space.

If $A = (a_{nk})$ is an infinite matrix with scalar entries the convergence domain

$$
c_A = \left\{ x \in \omega \mid Ax = \left( \sum_{k} a_{nk} x_k \right)_{n=1}^{\infty} \in c \right\}
$$
admits a natural $FK$-topology [21]. For $x \in \mathcal{L}_A$ we write $\lim_{n} x = \lim_{A} x$.

If $\varphi \subset \mathcal{L}_A$ let $a_n = \lim_{n} a_{nk}$ and define

$$I_A = \left\{ x \in \mathcal{L}_A \mid \sum_{k} a_k x_k \text{ exists} \right\}.$$  

$$\Lambda_A : I_A \to \mathbb{C} \text{ by } \Lambda_A(x) = \lim_{A} x - \sum_{k} a_k x_k \text{ (where } \mathbb{C} \text{ or } \mathbb{R} \text{) and}$$

$$\Lambda_A = \left\{ x \in I_A \mid \Lambda_A(x) = 0 \right\}.$$  

Further if $\varphi \subset \mathcal{L}_A$ we write $I_{\mathcal{L}_A}, W_{\mathcal{L}_A}, S_{\mathcal{L}_A}$ instead of $I_{\mathcal{L}_A}, W_{\mathcal{L}_A}, S_{\mathcal{L}_A}$. In this case $W_{\mathcal{L}_A} = L_{\mathcal{L}_A} \cap \Lambda_A^\perp$ (see e.g. [24]).

3. THE GLIDING HUMP PROPERTIES.

We begin by introducing several types of gliding hump properties.

DEFINITION 3.1. A sequence $(y^{(n)})$ in $\omega \setminus \{0\}$ is called a block sequence if there exists an index sequence $(k_j)$ such that $y^{(n)} = 0$ for any $n, k \in \mathbb{N}$ with $k \not\subset \{k_{n-1}, k_n\}$ where $k_0 := 0$, and it is called a 1-block sequence if furthermore $y_k = 1$ for each $k \in \{k_{n-1}, k_n\}$ and $n \in \mathbb{N}$.

Let $E$ be a sequence space containing $\varphi$.

- $E$ has the gliding hump property (ghp) if for each block sequence $(y^{(n)})$ satisfying $\sup_{n \in \mathbb{N}}\|y^{(n)}\|_b < \infty$ and any monotonically increasing sequence $(n_k)$ of integers there exists a subsequence $(m_k)$ of $(n_k)$ with $\sum_{j=1}^{\infty} y^{(m_j)} \in E$ (pointwise sum).

- $E$ has the pointwise gliding hump property (p_ghp) if for each $x \in E$, any block sequence $(y^{(n)})$ satisfying $\sup_{n \in \mathbb{N}}\|y^{(n)}\|_b < \infty$ and any monotonically increasing sequence $(n_k)$ of integers there exists a subsequence $(m_k)$ of $(n_k)$ with $\sum_{j=1}^{\infty} y^{(m_j)} \in E$ (pointwise sum).

- $E$ has the uniform gliding hump property (u_ghp) if the sequence $(n_k)$ in the definition of the p_ghp may be chosen independently of $x \in E$.

- $E$ has the pointwise weak gliding hump property (p_wghp) if the definition of the p_ghp is fulfilled for each 1-block sequence.

- $E$ has the uniform weak gliding hump property (u_wghp) if the definition of the u_ghp is fulfilled for each 1-block sequence.

We say that $E$ has the strong p_ghp (u_ghp, p_wghp or u_wghp) if $\sum_{j=1}^{\infty} x y^{(m_j)} \in E$ (pointwise sum) holds for any subsequence of $(m_k)$ in the above definitions; in this case, we use the notation sp_ghp, su_ghp, sp_wghp and su_wghp, respectively.

REMARKS 3.2. Let $E$ be a sequence space containing $\varphi$.

(a) Obviously, the definition of the ghp corresponds with the definition given in [20],[4] and the definition of the p_wghp corresponds to the weak gliding hump property considered by D. Noll [14].

(b) $E$ has the u_ghp if and only if $\mathcal{M}(E)$ has the ghp.

(c) $su_\text{ghp} \Rightarrow su_\text{wghp} \Rightarrow u_\text{wghp} \Rightarrow p_\text{wghp};$

$\text{su_ghp} \Rightarrow \text{sp_ghp} \Rightarrow \text{sp_wghp} \Rightarrow \text{p_wghp};$

$\text{su_ghp} \Rightarrow \text{u_ghp} \Rightarrow \text{p_ghp} \Rightarrow \text{p_wghp};$


\[ su_{wghp} \Rightarrow sp_{wghp} \quad \text{and} \quad sp_{ghp} \Rightarrow p_{ghp}. \]

(In the last section we provide examples to show that most of these implications are strict.)

(d) Each solid space has the \( sp_{ghp} \) and each monotone space has the \( su_{wghp} \). (Note, each solid sequence space is monotone.)

(e) Examples of spaces \( E \) such that \( M(E) \) has the ghp may be found in [1, Remark 1].

(f) In [4] T. Leiger and the first author proved the validity of theorems of Mazur Orlicz type under the assumption that \( M \) is a sequence space such that \( M(M) \) has the ghp, that is, \( M \) has the \( u_{ghp} \). Actually, in each instance the fact that \( M(M) \) has the \( p_{ghp} \) was used in the arguments.

**THEOREM 3.3.** Let \( E \) be an \( FK \) space containing \( \varphi \). Then \( S_E \) has the strong \( p_{ghp} \); in particular, if \( E \) is an \( FK \cdot AK \) space then \( E \) has the strong \( p_{ghp} \).

**PROOF.** The \( FK \) topology of \( E \) may be generated by seminorms

\[ p_r \quad (r \in \mathbb{N}) \quad \text{such that} \quad p_r(x) \leq p_{r+1}(x) \quad (r \in \mathbb{N} \text{ and } x \in E). \]  

Since \( S_E \) is an \( FK \cdot AK \) space we may assume that \( E \) is an \( FK \cdot AK \)-space.

Now, let \( x \in E \) be given. Then

\[ \lim_{n \to \infty, \ r \in \mathbb{N}} p_r \left( \sum_{k=n}^{n+1} x_k e^k \right) = 0 \quad (r \in \mathbb{N} \text{ and } n \to \infty). \]  

Further let \( (y^{(j)}) \) be a subsequence of any block sequence satisfying \( M := \sup_{r \in \mathbb{N}} \| y^{(j)} \|_{L^r} < \infty \). There exist index sequences \( (\nu_j) \) and \( (\mu_j) \) such that \( \nu_j \leq \mu_j < \nu_{j+1} \quad (j \in \mathbb{N}) \) and

\[ y^{(j)} = \sum_{k=\nu_j}^{\mu_j} y^{(j)} e^k, \quad \text{thus} \quad y^{(j)} = 0 \quad \text{for} \quad k \notin [\nu_j, \mu_j]. \]

On account of (o) it is sufficient to prove \( xy^{(j)} \to 0 \) in \( E \). For that end let \( r \in \mathbb{N} \) be given. Then we have

\[ p_r(xy^{(j)}) = p_r \left( \sum_{k=\nu_j}^{\mu_j} x_k y^{(j)} e^k \right) \leq \sup_{K \geq \nu_j} p_r \left( \sum_{k=\nu_j}^{K} x_k e^k \right) \sum_{k=\nu_j}^{\mu_j} \left| y^{(j)} - y^{(j)} \right| \]

\[ \leq M \sup_{K \geq \nu_j} p_r \left( \sum_{k=\nu_j}^{K} x_k e^k \right) \to 0 \quad \text{as} \quad j \to \infty, \]

by (o) which proves \( xy^{(j)} \to 0 \) in \( E \).

**REMARK 3.4.** In general, \( W_E \) fails the \( p_{wghp} \). [Example: Let \( \Sigma \) be the summation matrix and \( E := c_{00} \). Then \( W_E \) fails the \( p_{wghp} \) since \( x := e = \sum e^l \in W_E \) (pointwise sum) and \( (n_j) = (2j) \) does not have any subsequence \( (m_j) \) such that \( x := \sum e^{m_j} \) (pointwise sum) \( \in E \) since \( \Sigma^{-1} \notin m_0 \setminus c. \]

**THEOREM 3.5.** Let \( E \) be a sequence space containing \( \varphi \), and let \( B \) be a matrix such that \( E \subset c_B \).

Then \( E \subset S_B \) if \( E \) has the \( p_{wghp} \).

**PROOF.** Suppose \( E \) has the \( p_{wghp} \). We know from Theorem 6 of D. Noll [14] and Remark 3.2(a) that \( (E^0, \sigma(E^0, E)) \) is weakly sequentially complete. Therefore, by an inclusion theorem of G. Bennett and N. J. Kalton [2, Theorem 5] we get \( E \subset W_B \), in particular \( E \subset L_B \) and \( E \subset A_B \).

Now, assume \( E \subset W_B \) and \( E \notin S_B \), that is, there exists an \( x \in E \subset W_B = L_B \cap A_B \) with \( x \notin S_B \), thus

\[ \lim_{n \to \infty} \sup_{r \in \mathbb{N}} \left| \sum_{k=1}^{r} b_k x_k \right| < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} b_k x_k \right| \to 0. \]
Therefore we may choose an $\eta > 0$ and index sequences $(\alpha_j), (\beta_j)$ and $(n_j)$ with $\alpha_j \leq \beta_j$ such that

$$\left| \sum_{k=\alpha_j}^{n_j} b_k x_k \right| \geq \eta \quad \text{for each } j \in \mathbb{N}.$$ 

Now we employ a gliding hump argument. Let $k_0 := 1$ and choose $n_1^*$ such that

$$\sum_{k=1}^{n_1^*} \left| b_k - b_{k_0} \right| |x_k| < 2^{-1} \quad (n \geq n_1^*).$$ 

Then there exist a $j_1 \in \mathbb{N}$ with $n_{j_1} > n_1^*$ and a $k_1 > \beta_{j_1}$ such that (note $x \in I_B$)

$$\max_{K \geq k_1 \in \mathbb{N}_0} \left\{ \left| \sum_{k=K}^{K+p} b_{k} x_k \right|, \left| \sum_{k=K}^{K+p} b_k z_k \right| \right\} < 2^{-1} \quad (n \leq n_{j_1}).$$ 

Choose $n_2^* \geq n_{j_1}$ such that

$$\sum_{k=1}^{n_2^*} \left| b_{n_k} - b_k \right| |x_k| < 2^{-2} \quad (n \geq n_2^*)$$

and a $k_2 > k_1$ such that

$$\max_{K \geq k_2 \in \mathbb{N}_0} \left\{ \left| \sum_{k=K}^{K+p} b_{n_k} x_k \right|, \left| \sum_{k=K}^{K+p} b_k z_k \right| \right\} < 2^{-2} \quad (n \leq n_2^*).$$

Proceeding inductively, we get index sequences $(k_r), (j_r), (n_r)$ with

$$n_1^* < n_{j_1} < n_2^* < n_3^* < n_{j_2} < \ldots < n_{2
u-1}^* < n_{j_\nu} < n_{2
u}^* < n_{2
u+1}^* < \ldots$$

and

$$k_0 < \alpha_{j_1} \leq \beta_{j_1} < k_1 < \alpha_{j_2} \leq \beta_{j_2} < \ldots < n_{2
u-2}^* < \alpha_{j_\nu} \leq \beta_{j_\nu} < k_{2
u-1} < k_{2\nu} < \ldots$$

fulfilling

$$\sum_{k=1}^{n_r} \left| b_{n_k} - b_k \right| |x_k| < 2^{-r} \quad (n \geq n_r^*).$$

and

$$\max_{K \geq k_r \in \mathbb{N}_0} \left\{ \left| \sum_{k=K}^{K+p} b_{n_k} x_k \right|, \left| \sum_{k=K}^{K+p} b_k z_k \right| \right\} < 2^{-r} \quad \begin{cases} \text{if } \nu \text{ is odd and } n \leq n_{j_\nu} & \text{if } \nu \text{ is even and } n \leq n_{2\nu}^* \end{cases}.$$

Now, we define a subsequence $(y(\nu))$ of a 1-block sequence by

$$y_k(\nu) := \begin{cases} 1 & \text{if } \alpha_{j_\nu} \leq k \leq \beta_{j_\nu} \\ 0 & \text{otherwise} \end{cases}$$

and consider

$$yz \quad \text{where } y := \sum_{k=1}^{\infty} y(\nu) \text{ (pointwise sum).}$$

Since $E$ has the p.wghp we may assume that $yz \in E$ (otherwise we switch over to a subsequence $(y(n^*)$ and adapt the chosen index sequences). For a proof of Theorem 3.5 it is sufficient to prove $yz = : x \notin c_B$ .

For this let $\nu \geq 2$ and $n := n_{j_\nu}$. Then (note, $\sum_{k} b_k z_k$ exists)

$$\left| \sum_{k=1}^{\infty} b_{n_k} z_k - \sum_{k=1}^{\infty} b_k z_k \right|$$

$$\geq \sum_{k=1}^{\infty} \left| b_{n_k} - b_k \right| |x_k| + \sum_{k=\alpha_{j_\nu}}^{n_{j_\nu}} b_{n_k} x_k - \sum_{r=\nu+1}^{\infty} \sum_{k=\alpha_{j_\nu}}^{n_{j_\nu}} b_{n_k} z_k$$

$$\geq \eta - 2^{-\nu} - 2^{-\nu} \longrightarrow \eta > 0 \text{ for } \nu \to \infty.$$
Now let $\nu \geq 2$ and $\nu:=\nu^*_\nu$. Then (see above)
\[\left| \sum_{k=1}^{\infty} b_{n_k} z_k - \sum_{k=1}^{\infty} b_k z_k \right| \leq \sum_{k=1}^{\infty} \left| b_{n_k} - b_k \right| \| z_k \| + \sum_{r=\nu+1}^{\infty} \left| \sum_{k=1}^{\nu} b_{n_k} x_k \right| \leq 2^{-\nu} + 2^{-r} \rightarrow 0 \quad (\nu \rightarrow \infty).\]

Altogether we have proved $y \notin c_B$. □

Now, an obvious question is whether the statement in Theorem 3.5 remains true if we replace the domain $c_B$ by any separable FK-space $F$ with $E \subset F$. A positive answer is a consequence of the following theorem.

**THEOREM 3.6.** Let $E$ be a sequence space containing $\varphi$. Then the following statements are equivalent:

(i) $(E, \tau(E, E^\beta))$ is an AK-space and $E^\beta$ is $\sigma(E^\beta, E)$-sequentially complete.

(ii) If $F$ is any separable FK-space with $E \subset F$ then $E \subset S_F$.

(iii) If $A$ is any matrix with $E \subset c_A$ then $E \subset S_A$.

**PROOF.** The equivalence (i)$\Leftrightarrow$(ii) is Theorem 6, (i)$\Leftrightarrow$(ii) of G. Bennett and N. J. Kalton [2]. The implication (ii)$\Rightarrow$(iii) is obviously valid since domains $c_A$ are separable FK-spaces.

We are going to prove (iii)$\Rightarrow$(i). Let (iii) be valid. Then $E^\beta$ is $\sigma(E^\beta, E)$-sequentially complete by [2, Theorem 5, (iv)$\Rightarrow$(i)].

Assume, $(E, \tau(E, E^\beta))$ is not AK. Thus, we may choose an $x \in E$ and an absolutely convex $\sigma(E^\beta, E)$-compact subset $K$ of $E$ such that
\[p_K(x^{[n]} - x) \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{where} \quad p_K(z) := \sup_{a \in K} \left| \sum_{k} a_k z_k \right| \quad (z \in E).\]

Therefore we may choose an index sequence $(n_k)$ and a sequence $(a^{(i)})$ in $K$ such that
\[\left| \sum_{k=n_{i+1}}^{\infty} a^{(i)}_k z_k \right| \geq \eta > 0 \quad (i \in \mathbb{N}). \quad (*)\]

Since $K$ is $\sigma(E^\beta, E)$-compact, $\sigma(E^\beta, E)$ and $\sigma(E^\beta, \varphi)$ coincide on $K$ and $\sigma(E^\beta, \varphi)$ is metrizable we may assume that $(a^{(i)})$ is $\sigma(E^\beta, E)$-convergent to an $a \in K$. (Otherwise we switch over to a subsequence of $(a^{(i)})$.) If $A$ denotes the matrix given by
\[a_{ik} := a^{(i)}_k \quad (i, k \in \mathbb{N})\]
then—in summability language—the last assumption tells us
\[E \subset c_A \quad \text{(even $E \subset A^+_A$)}.\]

From $(*)$ we get $x \notin S_A$ which contradicts the assumption that (iii) is true. □

**COROLLARY 3.7.** Let $E$ be a sequence space containing $\varphi$ and $F$ be a separable FK-space with $E \subset F$. If $E$ has the p.wghp then $E \subset S_F$.

**PROOF.** Theorem 3.6 and 3.5. □

**COROLLARY 3.8.** Let $Y$ be a sequence space and $E$ be an FK-space with $\varphi \subset Y \cap E$ and $B$ be a matrix with $Y \cap S_E \subset c_B$. Then $Y \cap S_E \subset S_B$ if $Y$ has the p.wghp.

The statement remains true if we replace $c_B$ by any separable FK-space $F$. 
Proof. Corollary 3.7 and the fact that $Y \cap S_E$ has the p.wghp.

Corollary 3.9. Let $E$ be a separable FK space containing $\varphi$ such that $S_E \not\subseteq W_E$. Then $W_E$ fails the p.wghp (whereas $S_E$ has the strong p.ghp).

Proof. Theorem 3.3 and Corollary 3.7.

4. Applications.

Remark 4.1. Let $A = (a_{nk})$ be a matrix with $\varphi \subseteq c_A$ and let $x \in c_A$. Then

$$x \in S_A \iff \sum_{k=1}^{\infty} a_{nk} x_k \text{ converges uniformly in } n \in \mathbb{N}.$$

This observation gives us a short proof of the following theorem containing a Toeplitz-Silverman theorem.

Theorem 4.2 (matrices being conservative for $c_0$). For matrices $A = (a_{nk})$ the following statements are equivalent:

(a) $c_0 \subseteq c_A$.

(b) $c_0 \subseteq S_A$.

(c) $c_0 \subseteq c_A$ and $\|A\| := \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

Proof. The implication (a) $\Rightarrow$ (b) comes from the AK-property of $c_0$ and the monotonicity of FK-topologies. (This statement follows also by Theorem 3.5 since $c_0$ obviously has the p.wghp.) Using standard estimations we may prove (c) $\Rightarrow$ (a). We are going to prove the essential part (b) $\Rightarrow$ (c).

Let $c_0 \subseteq S_A$. Therefore, we can apply the above remark to any $x \in c_0$.

If $\|A\| = \infty$ we may choose a sequence $(n_j)$ in $\mathbb{N}$ and index sequences $(\alpha_j)$ and $(\beta_j)$ with $\alpha_j \leq \beta_j < \alpha_{j+1}$ ($j \in \mathbb{N}$) such that

$$\sum_{k=\alpha_j}^{\beta_j} |a_{n_k,k}| \geq j^2 \quad (j \in \mathbb{N}).$$

Defining $y \in c_0$ by

$$y_k := \begin{cases} \frac{1}{j} \text{sgn} a_{n_k,k} & \text{if } \alpha_j \leq k \leq \beta_j \\ 0 & \text{otherwise} \end{cases}$$

we get

$$\left| \sum_{k=\alpha_j}^{\beta_j} a_{n_k,k} y_k \right| = \frac{1}{j} \sum_{k=\alpha_j}^{\beta_j} |a_{n_k,k}| \geq j \quad (j \in \mathbb{N}).$$

Thus $\sum_{k=1}^{\infty} a_{nk} y_k$ does not converge uniformly in $n \in \mathbb{N}$ which contradicts $c_0 \subseteq S_A$.

Using the same method we get also a proof of a theorem containing a theorem of Hahn (equivalence of (a) and (c)). However, we should mention that the proof of (a) $\Rightarrow$ (c) presented in [18, Theorem 4.1, p. 110] is more elegant.

Theorem 4.3 (matrices summing each absolute summable sequence). For matrices $A = (a_{nk})$ the following statements are equivalent:

(a) $\ell \subseteq c_A$.

(b) $\ell \subseteq S_A$.

(c) $\varphi \subseteq c_A$ and $\sup_{n,k \in \mathbb{N}} |a_{nk}| < \infty$. 

PROOF. (a) \(\Rightarrow\) (b) follows from the continuity of the inclusion map and the fact that \(\ell\) is an FK-\(\mathcal{L}\)K-space and the monotonicity of FK-topologies whereas (c) \(\Rightarrow\) (a) may be proved with classical estimations. (b) \(\Rightarrow\) (c): Let \(\ell \subset S_A\). Thus, obviously, \(\varphi \subset c_A\) is true. We assume \(\sup_{n \in \mathbb{N}} |a_{nk}| = \infty\). Then we may choose sequences \((n_j)\) and \((k_j)\) in \(\mathbb{N}\) with \(k_j < k_{j+1}\) \((j \in \mathbb{N})\) such that

\[|a_{nk}| \geq j^2 \quad (j \in \mathbb{N})\]

Defining \(y = (y_k)\) by

\[y_k = \begin{cases} \frac{1}{j^2} \text{sgn} a_{nk}, & \text{if } k = k_j \\ 0 & \text{otherwise} \end{cases}\]

we obviously get

\[\left| \sum_{k=k_j}^{k_{j+1}} a_{nk} y_k \right| = \frac{1}{j^2} |a_{nk}| \geq 1 \quad (j \in \mathbb{N}).\]

The last estimation gives us \(y \notin S_A\) that contradicts \(\ell \subset S_A\). \(\square\)

In the next step we use this method to reprove both the well-known Schur theorem and the Hahn theorem. (The Schur theorem characterizes the matrices summing all bounded sequences, the Hahn theorem tells us that a conservative matrix which sums all \(x \in \chi\) sums also all bounded sequences where \(\chi\) denotes the set of all sequences with 0 and 1.) Moreover, we take an extended version of Schur's theorem (see [3]) into consideration.

**THEOREM 4.4 (Extended theorem of Schur, theorem of Hahn).** Let \(A = (a_{nk})\) be a matrix. Then the following statements are equivalent:

(a) \(m \subset c_A\).

(a*) \(m \subset S_A\).

(b) \(\exists \mu = (\mu_k), 0 < \mu_k \not\to \infty: m_\mu \subset c_A\).

(b*) \(\exists \mu = (\mu_k), 0 < \mu_k \not\to \infty: m_\mu \subset S_A\).

(c) \(\chi \subset c_A\), that is \(m_0 \subset c_A\).

(c*) \(\chi \subset S_A\), that is \(m_0 \subset S_A\).

(d) \(\varphi \subset c_A\) and \(\limsup \sum_{k=1}^{\infty} |a_{nk} - a_k| = 0\) where \(a_k\) denotes the limit of the \(k\)-th column.

(d*) \(c_0 \subset c_A\) and \(\limsup \sum_{k=1}^{\infty} \mu_k |a_{nk} - a_k| = 0\) where \(\mu_k\) is a common sequence \(\mu\).

(\(e\)) \(\mu \subset c_A\) and \(\exists \mu = (\mu_k), 0 < \mu_k \not\to \infty: \limsup \sum_{k=1}^{\infty} \mu_k |a_{nk} - a_k| = 0\).

Thereby, we can choose in (b), (b*) and (e) a common sequence \(\mu\).

**REMARK 4.5.** Originally, Schur proved \'(a) \Leftrightarrow (d)^*\' and \(\lim_{n \to \infty} x = \sum_{k=1}^{\infty} a_k z_k (x \in m)\) if (a) or (d*) in 4.4 is valid.

In case of conservative matrices the equivalence (a) \(\Leftrightarrow\) (e) is Hahn's theorem.

**PROOF of 4.4.** We are going to check the following chain of implications:

\[\text{(b)} \overset{(1)}{\Rightarrow} \text{(b*}) \overset{(3)}{\Rightarrow} \text{(a*}) \overset{(4)}{\Rightarrow} \text{(a}) \overset{(5)}{\Rightarrow} \text{(c*}) \overset{(6)}{\Rightarrow} \text{(d}) \overset{(7)}{\Rightarrow} \text{(d*}) \overset{(8)}{\Rightarrow} \text{(e*}) \overset{(9)}{\Rightarrow} \text{(e}) \overset{(10)}{\Rightarrow} \text{(b}).\]

The implications (2), (3) and (4) and the equivalences (7) and (9) are obviously true.

The implications (1) and (5) are immediate corollaries of Theorem 3.5 since \(m_\mu\) and \(m_0\) have the p.wghp.
For a proof of (8) and (10) we refer to [3].

Now, we give a proof of (6). For that we assume that \( A \) is a matrix with real entries. [In the general case of complex entries we have to note that \( \sum_k |a_{nk}| \) converges uniformly in \( n \in \mathbb{N} \) if and only if this is true for the real part of \( a_{nk} \) and the imaginary part of \( a_{nk} \).

Let \( (e^*) \) be true. Then \( \mathcal{E} \subset c_A \).

If \( \sum_{k=1}^{\infty} |a_{nk}| \) does not converge uniformly in \( n \in \mathbb{N} \) then we may choose an \( \eta > 0 \), a sequence \( (n_j) \) in \( \mathbb{N} \) and index sequences \( (\alpha_j) \) and \( (\beta_j) \) with \( \alpha_j \leq \beta_j < \alpha_{j+1} \) such that

\[
\sum_{k=\alpha_j}^{\beta_j} |a_{nk}| \geq \eta \quad (j \in \mathbb{N}).
\]

We define \( y \in m_0 \) by

\[
y_k := \begin{cases} 
\text{sgn} a_{n_k} & \text{if } \alpha_j \leq k \leq \beta_j \\
0 & \text{otherwise}
\end{cases}
\]

Since

\[
|\sum_{k=\alpha_j}^{\beta_j} a_{nk} y_k| = \sum_{k=\alpha_j}^{\beta_j} |a_{nk}| \geq \eta \quad (j \in \mathbb{N})
\]

the series \( \sum_{k=1}^{\infty} a_{nk} y_k \) does not converge uniformly in \( n \in \mathbb{N} \). Therefore \( y \notin S_A \) which contradicts \( \mathcal{E} \subset S_A \). 

5. EXAMPLES.

The aim of this section is the presentation of some examples distinguishing almost all of the gliding hump properties. For that purpose we collect known connections between gliding hump and related properties of sequence spaces in the following graphic.

![Diagram showing connections between gliding hump properties](image-url)

Figure 1:
Each arrow stands for 'implies' and the corresponding number in the circle gives the number of the example in 5.1 proving the strictness of the implication.
EXAMPLES 5.1. (1) \( m_n \) is a monotone space, thus it has all of the weak gliding hump properties \( \text{wghp} \). However, it does not have the \( \text{ghp} \), thus no \( \text{wghp} \) and it is not solid.

(2) The sequence space \( f_0 \) of all sequences almost convergent to 0 has all of the gliding hump properties. Furthermore, it is not a monotone space.

For a proof of the first statement we may prove that \( f_0 \) has the \( \text{ghp} \) by modifying Snyders proof of [20, Theorem 7].

(3) \( E := \ell_2 \cap (c_n)_l \), with \( \mu_k = k^3 \) \((k \in \mathbb{N})\), has the \( \text{u.ghp} \) since \( \mathcal{M}(E) \) has the gliding hump property (see [23, Theorem 3.3 and 3.1]). We don't know whether \( E \) has the \( \text{su.ghp} \). Therefore, it may be a candidate to distinguish the properties \( \text{su.ghp} \) and \( \text{ghp} \).

(4a) \( E := \ell_2 \cap c_0 \) has the \( \text{sp.ghp} \) (thus all \( \text{?p.ghp} \)) since it is an \( \text{FK-AM} \) space (Theorem 3.3). Furthermore, with [17, Corollary 4.4] we get that \( E \) is a sum space. Thus, by definition of a sum space \( \mathcal{M}(E) := E' \).

Therefore, \( \mathcal{M}(E) = E' = \ell_2 + cs^l = \ell_2 + b \subseteq c \). From this and the fact that \( c \in \mathcal{M}(E) \) we may derive that \( E \) cannot have the \( \text{wghp} \) (thus \( \text{?u.ghp} \)).

(4b) Considering the James space we get further sequence spaces having the same gliding humps properties as the example in (4a). For that let \( \omega \) be the space of all real sequences and let

\[
N(x) = \sup \left\{ \sum_{i=1}^{n} (x_{p_{2m-1}} - x_{p_{2m}})^2 + x_{p_{2m+1}}^2 \right\}^{1/2}
\]

where the supremum is taken over all positive integers \( n \) and all finite increasing sequences of integers \( p_1, \ldots, p_{2m+1} \). Then

\[
S_N = \left\{ x \in \omega \ \middle| \ N(x) < \infty \right\}
\]

(together with its natural norm \( N \)) is a \( BK \)-space and the closure \( J = S_N^\infty \) of \( \varphi \) in \( S_N \) is called James space (see [16]). We'll make use of the following facts:

(i) \( S_N \) is a \( BK \)-algebra with identity \( e \).

(ii) \( J = S_N \cap e_0 \).

(iii) \( (e^k, E_k) \) is a shrinking basis for \( J' \) so, in particular, \( J' \) is \( AK \) thus \( AD \).

(iv) \( S_N = J \oplus (e) = J' \).

Now by (i) and (iv) we get \( \mathcal{M}(J') = J' \) and by (iii) we get \( \mathcal{M}(J') = \mathcal{M}(J'') \) (see [7, Proposition 3.4]). Therefore by (ii) and (iv) we have

\[
\mathcal{M}(J'') = \mathcal{M}(J'') = J'' = J \oplus (e) \subseteq c.
\]

As in (4a) we conclude that \( J' \) has the \( \text{sp.ghp} \) (thus all \( \text{?p.ghp} \)) since it is an \( \text{FK-AM} \)-space. Furthermore, by (\( \ast \)) and \( c \in \mathcal{M}(J') \) we get that \( J' \) cannot have the \( \text{u.ghp} \) (thus \( \text{?u.ghp} \)).

From (\( \ast \)) we know that \( J' \) is a sum space. Thus by [17, Corollary 4.4] \( J' \cap E \) will be a sum space too if \( E \) is any \( FK \)-space with unconditional basis \( (e^k) \).

Then

\[
\mathcal{M}(J' \cap E) = (J' \cap E)' = J'' + E' = (J \oplus (e)) + E'
\]

Now, let \( E \) be any \( FK \)-space with unconditional basis \( (e^k) \) such that \( E' \subseteq c \). Then, as above we may conclude, \( J' \cap E \) has all \( \text{?p.ghp} \) and no \( \text{?u.ghp} \).
GLIDING HUMP PROPERTIES AND SOME APPLICATIONS

(5) Obviously, bs does not have the p-ghlp thus none of the gliding hump properties in consideration. However it is known that (bs, τ(bs, βn)) is an AK space and (βn, σ(βn, bs)) is sequentially complete.

(6) Let $F$ be any separable FK space with $S_F \subseteq W_F$ (for example, the domain of a conull matrix being not strongly conull) and let $E := W_F$. Then $(E', \sigma(E', E))$ is sequentially complete (see [1, Theorem 1 and 2]) but $(E, \tau(E, E'))$ is not an AK space since otherwise from Theorem 3.6 we would get $S_E \supseteq E = W_F$ thus $S_F = W_F$.

Closing the paper we mention, that we don’t know whether there is a difference between the $s^?ghlp$ and the corresponding $s^?ghlp$ (see Figure 1 and Example 5.1(3)).

REFERENCES


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