EXISTENCE THEOREMS FOR A SECOND ORDER
m-POINT BOUNDARY VALUE PROBLEM AT
RESONANCE

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Abstract

Let \( f : [0,1] \times \mathbb{R} \to \mathbb{R} \) be a function satisfying Caratheodory’s conditions and \( e(t) \in L^1[0,1] \). Let \( \eta \in (0,1), \xi_i \in (0,1), a_i \geq 0, i = 1,2, \ldots, m-2 \), with \( \sum_{i=1}^{m-2} a_i = 1 \), \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \) be given. This paper is concerned with the problem of existence of a solution for the following boundary value problems

\[
\begin{align*}
  x''(t) &= f(t, x(t), x'(t)) + e(t), 0 < t < 1, \\
  x'(0) &= 0, x(1) = x(\eta), \\
  x''(t) &= f(t, x(t), x'(t)) + e(t), 0 < t < 1, \\
  x'(0) &= 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i).
\end{align*}
\]

Conditions for the existence of a solution for the above boundary value problems are given using Leray Schauder Continuation theorem.

Keywords and Phrases: three-point boundary value problem, m-point boundary value problem, Leray Schauder Continuation theorem, Caratheodory’s conditions, Arzela-Ascoli Theorem.

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1 INTRODUCTION.

Let \( f : [0,1] \times \mathbb{R} \to \mathbb{R} \) be a function satisfying Caratheodory’s conditions, \( e : [0,1] \to \mathbb{R} \) be a function in \( L^1[0,1] \), \( a_i \geq 0, \xi_i \in (0,1), i = 1,2, \ldots, m-2 \) with \( \sum_{i=1}^{m-2} a_i = 1 \), \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \) and \( \eta \in (0,1) \) be given. We study the problem of existence of solutions for the following boundary value problems

\[
\begin{align*}
  x''(t) &= f(t, x(t), x'(t)) + e(t), 0 < t < 1, \\
  x'(0) &= 0, x(1) = x(\eta), \\
  x''(t) &= f(t, x(t), x'(t)) + e(t), 0 < t < 1, \\
  x'(0) &= 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i).
\end{align*}
\]

It is well-known, (see, e.g. [1]), that if \( x \in C^1[0,1] \) satisfies the boundary conditions in (2), with the \( a_i \)'s as above, then there exists an \( \eta \in [\xi_1, \xi_{m-2}] \), depending on \( x \in C^1[0,1] \), such that

\[
x(1) = x(\eta).
\]

Accordingly, it seems that one can study the problem of existence of a solution for the boundary value problem (2) using the a priori estimates obtained for the three-point boundary value problem (1), as it was done in [2], [3], [4]. But here the m-point boundary value problem (2) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

\[
\begin{align*}
  x''(t) &= 0, 0 < t < 1, \\
  x'(0) &= 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),
\end{align*}
\]

has \( x(t) = A \), \( A \in \mathbb{R} \), as a non-trivial solution, since \( \sum_{i=1}^{m-2} a_i = 1 \). The result is that \( e(t) \in L^1[0,1] \) has to be such that \( \sum_{i=1}^{m-2} a_i \int_0^1 (1-\zeta_i) e(s) ds + \int_0^1 (1-s) e(s) ds = 0 \), (in view of the nonlinear Fredholm
alternative), so even though there exists an \( \eta \in [\xi_1, \xi_{m-2}] \) such that 
\[
\int_0^1 (1-\eta) e(s)ds + \int_0^1 (1-s) e(s)ds = \sum_{i=1}^{m-2} a_i \int_0^1 (1-\xi_i) e(s)ds + \int_0^1 (1-s) e(s)ds = 0, \text{ since } \sum_{i=1}^{m-2} a_i = 1, \text{ this } \eta \text{ is not necessarily the same } \\
\eta \text{ as in (3). We are, accordingly, forced to study the m-point boundary value problem (2) directly and obtain results about the three-point boundary value problem (1) as a corollary to the results for the m-point boundary value problem. It is interesting to note that while in the nonresonance case we had to study the m-point boundary value problem, using the results for the three-point boundary value problem, it is just the reverse case in the resonance case.}
\]

We obtain conditions for the existence of a solution for the boundary value problem (2), using Mawhin’s version of the Leray Schauder Continuation theorem [5] or [6] or [7]. Recently, Gupta, Ntouyas and Tsamatos studied the m-point boundary value problem
\[
x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\
x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),
\]
with \( \xi_i \in (0,1), \) \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \) \( a_i \in R, \) all \( a_i \) having the same sign, given, and \( \sum_{i=1}^{m-2} a_i \neq 1, \) in [3]. The boundary value problem (2) differs from the boundary value problem (4) in that the associated linear boundary value problem with (2), namely,
\[
x''(t) = 0, \quad 0 < t < 1, \\
x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),
\]
has \( x(t) = A, \) for \( A \in R, \) as non-trivial solutions, since \( \sum_{i=1}^{m-2} a_i = 1, \) while the corresponding linear boundary value problem associated with (4), namely,
\[
x''(t) = 0, \quad 0 < t < 1, \\
x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),
\]
with \( \sum_{i=1}^{m-2} a_i \neq 1, \) has \( x(t) = 0, \) as its only solution. It is for this reason we call the boundary value problem (2) to be at resonance. For some recent results on m-point and three-point boundary value problems we refer the reader to [2], [3], [4], [8], [9], [10], and [11].

We use the classical spaces \( C^k[0,1], \) \( C^k[0,1], \) \( L^k[0,1], \) and \( L^\infty[0,1] \) of continuous, \( k \)-times continuously differentiable, measurable real-valued functions whose \( k \)-th power of the absolute value is Lebesgue integrable on \( [0,1], \) or measurable functions that are essentially bounded on \( [0,1], \) and we also use the Sobolev space \( W^{2,k}(0,1), \) \( k = 1,2 \) defined by
\[
W^{2,k}(0,1) = \{ x : [0,1] \to R \text{ with } x, x' \text{ abs. cont. on } [0,1] \text{ with } x'' \in L^k[0,1] \}
\]
with its usual norm. We denote the norm in \( L^k[0,1] \) by \( \| . \|_k, \) and the norm in \( L^\infty[0,1] \) by \( \| . \|_\infty. \)

2. EXISTENCE THEOREMS.

Let \( X, Y \) denote Banach spaces \( X = C^1[0,1] \) and \( Y = L^1[0,1] \) with their usual norms. Let \( Y_2 \) be the subspace of \( Y \) spanned by the function 1, i.e.
\[
Y_2 = \{ x(t) \in Y \mid x(t) = A, \text{ a.e. on } [0,1], A \in R \}
\]
and let \( Y_1 \) be the subspace of \( Y \) such that \( Y = Y_1 \oplus Y_2. \) Let \( a_i \geq 0, \xi_i \in (0,1), i = 1,2, \cdots, m-2 \) with \( \sum_{i=1}^{m-2} a_i = 1, \) \( 0 < \xi_1 < \xi_2 < \cdots, \xi_{m-2} < 1, \) be given. We note that for \( x(t) \in Y \) we can write
\[
x(t) = (x(t) - A) + A, \quad \text{ with } A = \sum_{i=1}^{m-2} a_i \int_0^1 (1-\xi_i) x(s)ds + \int_0^1 (1-s) x(s)ds,
\]
\( t \in [0,1], \) for \( t \in [0,1], \) we define the canonical projection operators \( P : Y \to Y_1, Q : Y \to Y_2 \) by
\[
P(x(t)) = x(t) - \sum_{i=1}^{m-2} a_i \int_0^1 (1-\xi_i) x(s)ds + \int_0^1 (1-s) x(s)ds,
\]
\[
Q(x(t)) = \sum_{i=1}^{m-2} a_i \int_0^1 (1-\xi_i) x(s)ds + \int_0^1 (1-s) x(s)ds,
\]
for \( x(t) \in Y. \) We note that if \( Q(x(t)) = 0, \) there exists a \( \zeta \in (0,1) \) such that \( x(\zeta) = 0. \) Clearly, \( Q = I - P, \) where \( I \) denotes the identity mapping on \( Y, \) and the projections \( P \) and \( Q \) are continuous. Now let \( X_2 = X \cap Y_2. \) Clearly \( X_2 \) is a closed subspace of \( X. \) Let \( X_1 \) be the closed subspace of \( X \) such
that \( X = X_1 \oplus X_2 \). We note that \( P(X) \subseteq X_1, Q(X) \subseteq X_2 \) and the projections \( P \mid X : X \to X_1 \) and \( Q \mid X : X \to X_2 \) are continuous. In the following, \( X, Y, P, Q \) will refer to the Banach spaces and the projections as defined and we shall not distinguish between \( P, P \mid X \) (resp. \( Q, Q \mid X \)) and depend on the context for the proper meaning.

Define a linear operator \( L : D(L) \subseteq X \to Y \) by setting

\[
D(L) = \{ x \in W^{2,1}(0,1) \mid x'(0) = 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) \},
\]

and for \( x \in D(L) \),

\[
Lx = x''.
\]

Let, now, for \( e \in Y_1 \), i.e. \( e \in L^1[0,1] \) with \( \sum_{i=1}^{m-2} a_i \int_0^1 (1 - \xi_i) e(s)ds + \int_0^1 (1 - s) e(s)ds = 0 \), \( Ke \) denote the unique solution of the boundary value problem

\[
x''(t) = e(t), 0 < t < 1,
x'(0) = 0, x(1) = a_1 x(\xi_1),
\]

such that \( \sum_{i=1}^{m-2} a_i \int_0^1 (1 - \xi_i) x(s)ds + \int_0^1 (1 - s) x(s)ds = 0 \). Indeed, for \( t \in [0,1] \),

\[
(Ke)(t) = \int_0^1 (t-s)e(s)ds + \frac{A}{\sum_{i=1}^{m-2} a_i (1 - \xi_i)} [\int_0^1 (1 - \xi_i) x(s)ds + \int_0^1 (1 - s) x(s)ds - A] = 0.
\]

Accordingly the linear mapping \( K : Y_1 \to X_1 \) defined by the equation (12) is a bounded linear mapping and is such that for

\[
x \in Y, KPx \in D(L), \text{ and } LKP(x) = P(x).
\]

**DEFINITION 1** :- A function \( f : [0,1] \times R^2 \to R \) satisfies Caratheodory's conditions if (i) for each \( (x,y) \in R^2 \), the function \( f(t,x,y) \in R \) is measurable on \([0,1]\), (ii) for a.e. \( t \in [0,1] \), the function \( (x,y) \in R^2 \to f(t,x,y) \in R \) is continuous on \( R^2 \), and (iii) for each \( r > 0 \), there exists \( \alpha_r(t) \in L^1(0,1) \) such that \( |f(t,x,y)| \leq \alpha_r(t) \) for a.e. \( t \in [0,1] \) and all \( (x,y) \in R^2 \) with \( \sqrt{x^2 + y^2} \leq r \).

Let \( f : [0,1] \times R^2 \to R \) be a function satisfying Caratheodory's conditions. Let \( N : X \to Y \) be the non-linear mapping defined by

\[
(Nx)(t) = f(t,x(t),x'(t)), t \in [0,1],
\]

for \( x(t) \in X \).

For \( e(t) \in Y_1 \), i.e. \( e(t) \in L^1[0,1] \) with \( \sum_{i=1}^{m-2} a_i \int_0^1 (1 - \xi_i) e(s)ds + \int_0^1 (1 - s) e(s)ds = 0 \), the boundary value problem (2) reduces to the functional equation

\[
Lx = Nx + e,
\]

in \( X \), with \( e(t) \in Y_1 \), given.

**THEOREM 2** :- Let \( f : [0,1] \times R^2 \to R \) be a function satisfying Caratheodory's conditions. Assume that there exist functions \( p(t), q(t), r(t) \) in \( L^1(0,1) \) such that

\[
|f(t,x_1,x_2)| \leq p(t) |x_1| + q(t) |x_2| + r(t)
\]

for a.e. \( t \in [0,1] \) and all \( (x_1,x_2) \in R^2 \). Also let \( a_i \geq 0, \xi_i \in (0,1), i = 1,2, \ldots, m-2 \) with \( \sum_{i=1}^{m-2} a_i = 1, 0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1 \) be given, and assume that for every \( x(t) \in X \),

\[
(Qx)(t). (QNz)(t) \geq 0, \text{ for } t \in [0,1].
\]

Then for \( e(t) \in Y_1 \), i.e. \( e(t) \in L^1[0,1] \) with \( \sum_{i=1}^{m-2} a_i \int_0^1 (1 - \xi_i) e(s)ds + \int_0^1 (1 - s) e(s)ds = 0 \), given, the boundary value problem (2) has at least one solution in \( C^1[0,1] \) provided

\[
\|p\|_1 + \|q\|_1 < 1.
\]
PROOF:- We first note that the bounded linear mapping $K : Y_1 \to X_1$ defined by the equation (12) is such that the mapping $KPN : X \to X$ maps bounded subsets of $X$ into relatively compact subsets of $X$, in view of Arzela-Ascoli Theorem. Hence $KPN : X \to X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the boundary value problem (2) if and only if $x$ is a solution to the operator equation

$$Lx = Nx + e.$$  

Now, to solve the operator equation $Lx = Nx + e$, it suffices to solve the system of equations

$$Pz = KPNx + e_1,$$
$$QNz = 0,$$  

which gives on adding that $Lx = Nx + e$.

Now, (17) is clearly equivalent to the single equation

$$Pz + QNz - KPNx = e_1,$$  

which has the form of a compact perturbation of the Fredholm operator $P$ of index zero. We can, therefore, apply the version given in ([5], Theorem 1, Corollary 1) or ([6], Theorem IV.4) or ([7]) of the Leray-Schauder Continuation theorem which ensures the existence of a solution for (18) if the set of all possible solutions of the family of equations

$$Pz + (1 - \lambda)Qx + \lambda QNx - \lambda KPNx = \lambda e_1,$$  

is a priori bounded, independently of $\lambda$. Notice that (19) is then equivalent to the system of equations

$$Pz = \lambda KPNx + \lambda e_1,$$
$$(1 - \lambda)Qz + \lambda QNx = 0.$$  

Let, now, $x(t)$ be a solution of (20) for some $\lambda \in (0, 1)$. We see on multiplying the second equation in (20) and using (15) that $(1 - \lambda)((Qz)(t))^2 \leq 0$ for every $t \in [0,1]$. Hence $(Qz)(t) = 0$ for every $t \in [0,1]$ and accordingly there exists a $z(\zeta) = 0$. Since, now, $x'(0) = 0$ it follows that $||x||_{[a,b]} \leq ||x'||_{[a,b]} \leq ||x''||_{[a,b]}$. Also since $Qz = 0$, we have $QNz = 0$. It follows that $x \in D(L)$, i.e., $x \in W^{2,1}(0,1)$ with $x'(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\zeta_i)$ and $x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t)$. Accordingly, we get that

$$||x''||_1 = \lambda ||f(t, x(t), x'(t)) + e(t)||_1$$
$$\leq ||p||_1 ||x||_{[a,b]} + ||q||_1 ||x'||_{[a,b]} + ||r||_1 + ||e||_1$$
$$\leq (||p||_1 + ||q||_1) ||x''||_1 + ||r||_1 + ||e||_1$$

It follows from the assumption (16) that there is a constant $c$, independent of $\lambda \in (0, 1)$ and $x(t)$, such that

$$||x''||_1 \leq c.$$  

It is now immediate from $||x||_{[a,b]} \leq ||x'||_{[a,b]} \leq ||x''||_1$ that the set of solutions of the family of equations (20) is, a priori, bounded in $C^1[0,1]$ by a constant, independent of $\lambda \in (0, 1)$.

This completes the proof of the theorem.//

REMARK 1:- We remark that the Theorem 2 remains valid if we replace (15) by the condition

$$(Qz)(t)(QNz)(t) \leq 0, \text{ for } t \in [0,1].$$  

for every $x \in X$.

REMARK 2:- We remark that the condition (15) can be replaced by the condition

$$f(t, x_1, x_2)x_1 \geq 0,$$  

where $f(t, x_1, x_2) \in C^1[0,1]$.
for almost all $t \in (0,1)$ and all $(x_1, x_2) \in R^2$. Indeed, condition (15) was used to show, in the proof of Theorem 2, that if $x(t)$ is a solution of (20) for some $\lambda \in (0,1)$ then there exists a $\zeta \in (0,1)$ such that $x(\zeta) = 0$. We, now, show that (22), implies that if $x(t)$ is a solution of (20) for some $\lambda \in (0,1)$ then there exists a $\zeta \in (0,1)$ such that $x(\zeta) = 0$. Indeed, suppose that $x(t) \neq 0$, for all $t \in (0,1)$. We may, in fact, assume without any loss of generality that $x(t) > 0$, for every $t \in (0,1)$. It then follows from (22) that $f(t, x(t), x'(t)) \geq 0$, for a.e. $t \in (0,1)$. Hence $Qx > 0$ and $QN x \geq 0$. Now the second equation in (20) gives that $(1 - \lambda)(Qx)^2 + \lambda(QNx)(Qx) = 0$, so that we get $(Qx)^2 \leq 0$, a contradiction. Accordingly, there must exist a $\zeta \in (0,1)$ such that $x(\zeta) = 0$.

**THEOREM 3**: Let $f : [0, 1] \times R^2 \rightarrow R$ be a function as in Theorem 2. Assume that the functions $p(t), q(t), r(t)$ in (14) are in $L^2(0,1)$. Let $a_i \geq 0, \xi_i \in (0,1), i = 1, 2, \ldots, m - 2$ with $\sum_{i=1}^{m-2} a_i = 1, 0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$ be given.

Then for $e(t) \in L^2[0,1]$ with $\sum_{i=1}^{m-2} a_i \int_0^1 e(s)ds = 0$, given, the boundary value problem (2) has at least one solution in $C^1[0,1]$ provided

$$\frac{2}{\pi} \left( \frac{2}{\pi} ||p||_2 + ||q||_2 \right) < 1.$$  

**PROOF**: The proof is similar to the proof of Theorem 2, except now one uses the inequalities $||x||_2 \leq \frac{2}{\pi} ||x'||_2 \leq \frac{4}{\pi} ||x''||_2$ for an $x \in W^2[0,1]$ with $x(\zeta) = 0$, for some $\zeta \in (0,1)$ and $x'(0) = 0$ (see, Theorem 256 of [12]) to show that the set of solutions of the family of equations (19) is a priori bounded in $C^1[0,1]$ by a constant independent of $\lambda \in (0,1)$.

**THEOREM 4**: Let $f : [0, 1] \times R^2 \rightarrow R$ be a function as in Theorem 2 (respectively, Theorem 3). Let $\eta \in (0,1)$ be given. Then for $e(t) \in L^1[0,1]$ (respectively, $e(t) \in L^2[0,1]$) with $\int_0^1 (1 - \eta)e(s)ds + \int_0^1 (1 - s)e(s)ds = 0$, given, the three-point boundary value problem (1) has at least one solution in $C^1[0,1]$ provided

$$||p||_1 + ||q||_1 < 1,$$

(respectively, $\frac{2}{\pi} \left( \frac{2}{\pi} ||p||_2 + ||q||_2 \right) < 1$).

**PROOF**: The theorem follows immediately from Theorem 2 (respectively, Theorem 3) with $m = 3$ and $a_1 = 1, \xi_1 = \eta$.

**THEOREM 5**: Let $f : [0, 1] \times R^2 \rightarrow R$ be a function as in Theorem 2 (respectively, Theorem 3). Then for $e(t) \in L^1[0,1]$ (respectively, $e(t) \in L^2[0,1]$) with $\int_0^1 (1 - s)e(s)ds = 0$, given, the boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), 0 < t < 1,$$

$$x'(0) = 0, x(0) = x(1),$$

has at least one solution in $C^1[0,1]$ provided

$$||p||_1 + ||q||_1 < 1,$$

(respectively, $\frac{2}{\pi} \left( \frac{2}{\pi} ||p||_2 + ||q||_2 \right) < 1$).

**PROOF**: The theorem follows immediately from Theorem 2 (respectively, Theorem 3) with $m = 2$ and $a_1 = 1, \xi_1 = 0$.

**References**


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