CONVOLUTION OF HANKEL TRANSFORM AND ITS APPLICATION TO AN INTEGRAL INVOLVING BESSEL FUNCTIONS OF FIRST KIND

VU KIM TUAN

Institute of Mathematics, P.O.Box 631, Bo Ho, Hanoi, Vietnam

and

MEGUMI SAIGO

Department of Applied Mathematics, Fukuoka University, Fukuoka 814-01, Japan

(Received August 11, 1993)

Abstract

In the paper a convolution of the Hankel transform is constructed. The convolution is used to the calculation of an integral containing Bessel functions of the first kind.

Key Words and Phrases: Hankel transform, Bessel function, Convolution.

1991 AMS Subject Classification Codes: 33B20.

1. Introduction

The convolution of a modified Hankel transform, introduced in [4], has been studied in [1], [4] in classical sense and in [7] in a space of generalized functions. For an another modified Hankel transform the other convolution in some space of functions is obtained (see [5]).

The present paper is devoted to propose a definition of a convolution and to prove the convolution property in the classical sense of the following standard Hankel transform (see [6], [8])

$$\mathcal{H}_\nu[f](x) = \int_0^\infty y J_\nu(yx) f(y) dy, \quad \operatorname{Re}(\nu) > -\frac{1}{2}. \quad (1)$$

As one of its applications, a formula of infinite interval of a product of Bessel functions of the first kind is established.

2. Convolution of Hankel Transform

Set

$$h(x) = \frac{2^{1-3\nu} x^{-\nu}}{\sqrt{\pi} \Gamma(\nu + 1/2)} \iint_{u+v>x, |u-v|<x} [x^2 - (u-v)^2]^{\nu-1/2} [(u+v)^2 - x^2]^{\nu-1/2} \times (uv)^{1-\nu} f(u)g(v) du dv, \quad x \in (0, \infty). \quad (2)$$

The function $h(x)$ is called the Hankel convolution of the function $f(x)$ with the function $g(x)$. It is easy to see that the convolution is a commutative operator of f and g .

Let $L(R^+; \mu(x))$ be a class of integrable functions $f(x)$ with a weight $\mu(x) > 0$ in $R^+ = (0, \infty)$. The main aim of this section is to prove the following:

Theorem. Let $\operatorname{Re}(\nu) > \frac{1}{2}$ and $f(x), g(x) \in L(R_+; \sqrt{x})$. Then the function $h(x)$ in (2) exists and there holds the convolution property

$$\mathcal{H}_\nu[h](x) = x^{-\nu} \mathcal{H}_\nu[f](x) \mathcal{H}_\nu[g](x), \quad (3)$$

where $\mathcal{H}_\nu$ is the Hankel transform (1).

Proof. It is well known [6, (2.12.42.15)] that

$$\begin{aligned} & \int_0^\infty t^{1-\nu} J_\nu(xt) J_\nu(ut) J_\nu(vt) dt \\ &= \frac{2^{1-3\nu}}{\sqrt{\pi} \Gamma(\nu + 1/2)} (xuv)^{-\nu} \left[x^2 - (u-v)^2 \right]_+^{\nu-1/2} \left[(u+v)^2 - x^2 \right]_+^{\nu-1/2}, \end{aligned} \quad (4)$$

where $\operatorname{Re}(\nu) > -\frac{1}{2}$ and

$$\varphi_+(x) = \begin{cases} \varphi(x), & \varphi(x) \geq 0, \\ 0, & \varphi(x) < 0. \end{cases}$$

Since

$$J_\nu(x) = \begin{cases} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}), & (x \rightarrow +\infty), \\ O(x^\nu), & (x \rightarrow +0), \end{cases} \quad (5)$$

(see [3]) it is easy to conclude that there exists such a positive number C_1 independent of $x \in (0, \infty)$ that

$$|\sqrt{x} J_\nu(x)| < C_1, \quad x \in (0, \infty),$$

and $x^{-\nu} J_\nu(x) \in L(R_+)$ when $\operatorname{Re}(\nu) > \frac{1}{2}$. Therefore we have

$$\begin{aligned} \left| \int_0^N t^{1-\nu} J_\nu(xt) J_\nu(ut) J_\nu(vt) dt \right| &\leq \frac{C_1^2}{\sqrt{uv}} \int_0^N |t^{-\nu} J_\nu(xt)| dt \\ &\leq \frac{C_1^2 x^{\operatorname{Re}(\nu)-1}}{\sqrt{uv}} \int_0^\infty |t^{-\nu} J_\nu(t)| dt \leq \frac{C x^{\operatorname{Re}(\nu)-1}}{\sqrt{uv}}, \end{aligned} \quad (6)$$

where C is independent of x, u, v and N . In particular, making use the formulas (2) and (4) with the help of the estimate (6) we have

$$|h(x)| \leq C x^{\operatorname{Re}(\nu)-1} \int_0^\infty \int_0^\infty \sqrt{uv} |f(u)g(v)| du dv < \infty,$$

since $f(x), g(x) \in L(R_+; \sqrt{x})$. Thus the function $h(x)$ in (2) exists. Furthermore, applying the Fubini theorem, we obtain

$$\begin{aligned} h(x) &= \int_0^\infty \int_0^\infty uv f(u)g(v) \int_0^\infty t^{1-\nu} J_\nu(xt) J_\nu(ut) J_\nu(vt) dt du dv \\ &= \int_0^\infty t^{1-\nu} J_\nu(xt) \int_0^\infty \int_0^\infty uv J_\nu(ut) J_\nu(vt) f(u)g(v) du dv dt \\ &= \int_0^\infty t J_\nu(xt) t^{-\nu} \mathcal{H}_\nu[f](t) \mathcal{H}_\nu[g](t) dt. \end{aligned} \quad (7)$$

Here we have used the existence of the Hankel transform $\mathcal{H}_\nu$ defined by (1) for functions from $L(R_+; \sqrt{x})$ (see [2], [8]). Moreover, we notice the fact

$$\mathcal{H}_\nu[f](x) = O(x^\nu), \quad (x \rightarrow +0) \quad \text{for } f \in L(R_+; \sqrt{x})$$

from [2, p. 74]. Therefore, if we set

$$k(t) = t^{-\nu} \mathcal{H}_\nu[f](t) \mathcal{H}_\nu[g](t), \quad (8)$$

we have

$$k(t) = O(t^\nu), \quad (t \rightarrow +0). \quad (9)$$

On the other hand we have

$$|\mathcal{H}_\nu[f](t)| \leq \frac{1}{\sqrt{t}} \int_0^\infty |\sqrt{ut} J_\nu(ut) \sqrt{u} f(u)| du \leq \frac{C}{\sqrt{t}}, \quad t \in (0, \infty), \quad (10)$$

and therefore,

$$k(t) = O(t^{-\nu-1}), \quad (t \rightarrow +\infty). \quad (11)$$

Since $\operatorname{Re}(\nu) > 1/2$, from (9), (11) we conclude that $k(t) \in L(R_+; \sqrt{t})$. Therefore the formula (7) can be rewritten in the form

$$h(x) = \mathcal{H}_\nu[k](x).$$

Hence, by using the inversion formula of the Hankel transform in the class $L(R_+; \sqrt{x})$ (see [2], [8]):

$$\mathcal{H}_\nu[\mathcal{H}_\nu[k]](x) = k(x), \quad (12)$$

we obtain

$$k(x) = \mathcal{H}_\nu[h](x). \quad (13)$$

As $k(x)$ has the form (8), the formula (13) coincides with the formula (3). Thus the theorem is proved.

3. Application

As an application of Theorem we consider the integral

$$f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(a_0, a_1, \dots, a_n) = \int_0^\infty t^{\nu_0+1} J_{\nu_0}(a_0 t) \prod_{j=1}^n (t^2 + y_j^2)^{-\nu_j/2} J_{\nu_j}(a_j \sqrt{t^2 + y_j^2}) dt$$

with $a_j > 0$ ($j = 0, 1, \dots, n$) and $\operatorname{Re}(y_j) \geq 0$ ($j = 1, 2, \dots, n$). We will prove that

$$f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(a_0, a_1, \dots, a_n) = 0$$

when

$$a_0 > a_1 + \dots + a_n \quad \text{and} \quad \frac{1}{2} < \operatorname{Re}(\nu_0) < \sum_{j=1}^n \operatorname{Re}(\nu_j) + \frac{n-3}{2}.$$

We know that it is valid for $n = 1$ (see [6, (2.12.31.1)] for the case $\operatorname{Re}(y_1) = 0$, and [6, (2.12.35.12)] for the case $\operatorname{Re}(y_1) > 0$). Suppose that it is valid for every $k \leq n$. We have to prove it for the case $k = n + 1$. Put

$$g_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(t, a_1, \dots, a_n) = t^{\nu_0} \prod_{j=1}^n (t^2 + y_j^2)^{-\nu_j/2} J_{\nu_j}(a_j \sqrt{t^2 + y_j^2}).$$

By using (5) we have

$$\begin{aligned} g_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(t, a_1, \dots, a_n) &= O(t^{\nu_0}), & (t \rightarrow +0) \\ &= O(t^{\nu_0 - \nu_1 - \dots - \nu_n - n/2}), & (t \rightarrow +\infty). \end{aligned} \quad (14)$$

Suppose that

$$\frac{1}{2} < \operatorname{Re}(\nu_0) < \sum_{j=1}^n \operatorname{Re}(\nu_j) + \frac{n-3}{2}.$$

Then from (14) we conclude that $g_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(t, a_1, \dots, a_n) \in L(R_+; \sqrt{t})$. Therefore, by using the formula (10) we obtain

$$\mathcal{H}_{\nu_0} \left[g_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(t, a_1, \dots, a_n) \right] (x) = O\left(\frac{1}{\sqrt{x}}\right), \quad (x \rightarrow +0, x \rightarrow +\infty). \quad (15)$$

Since

$$\mathcal{H}_{\nu_0} \left[g_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(t, a_1, \dots, a_n) \right] (x) = f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(x, a_1, \dots, a_n),$$

the formula (15) can be read as

$$f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(x, a_1, \dots, a_n) = O\left(\frac{1}{\sqrt{x}}\right), \quad (x \rightarrow +0, x \rightarrow +\infty).$$

But by the assumption we have

$$f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(x, a_1, \dots, a_n) = 0$$

when $x > a_1 + \dots + a_n$. Therefore

$$f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(x, a_1, \dots, a_n) \in L(R_+; \sqrt{x})$$

and by (12)

$$\mathcal{J}\mathcal{C}_{\nu_0} \left[f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(x, a_1, \dots, a_n) \right] (t) = g_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(t, a_1, \dots, a_n).$$

Analogously, we have

$$f_{y_{n+1}}^{\nu_0, \nu_{n+1}}(x, a_{n+1}) \in L(R_+; \sqrt{x})$$

and

$$\mathcal{J}\mathcal{C}_{\nu_0} \left[f_{y_{n+1}}^{\nu_0, \nu_{n+1}}(x, a_{n+1}) \right] (t) = g_{y_{n+1}}^{\nu_0, \nu_{n+1}}(t, a_{n+1}).$$

under the conditions

$$\frac{1}{2} < \operatorname{Re}(\nu_0) < \operatorname{Re}(\nu_{n+1}) - 1.$$

Since

$$g_{y_1, \dots, y_{n+1}}^{\nu_0, \nu_1, \dots, \nu_{n+1}}(t, a_1, \dots, a_{n+1}) = t^{-\nu_0} g_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(t, a_1, \dots, a_n) g_{y_{n+1}}^{\nu_0, \nu_{n+1}}(t, a_{n+1}),$$

then by using the theorem we obtain

$$\begin{aligned} f_{y_1, \dots, y_{n+1}}^{\nu_0, \nu_1, \dots, \nu_{n+1}}(x, a_1, \dots, a_{n+1}) &= \int_0^\infty t J_{\nu_0}(xt) t^{-\nu_0} g_{y_{n+1}}^{\nu_0, \nu_{n+1}}(t, a_{n+1}) g_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(t, a_1, \dots, a_n) dt \\ &= \mathcal{J}\mathcal{C}_{\nu_0} \left[t^{-\nu_0} \mathcal{J}\mathcal{C}_{\nu_0} \left[f_{y_{n+1}}^{\nu_0, \nu_{n+1}}(y, a_{n+1}) \right] (t) \cdot \mathcal{J}\mathcal{C}_{\nu_0} \left[f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(y, a_1, \dots, a_n) \right] (t) \right] (x) \\ &= \frac{2^{1-3\nu_0} x^{-\nu_0}}{\sqrt{\pi} \Gamma(\nu_0 + 1/2)} \int \int_{u+v > x, |u-v| < x} (x^2 - (u-v)^2)^{\nu_0-1/2} ((u+v)^2 - x^2)^{\nu_0-1/2} (uv)^{1-\nu_0} \\ &\quad \times f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(u, a_1, \dots, a_n) f_{y_{n+1}}^{\nu_0, \nu_{n+1}}(v, a_{n+1}) du dv. \end{aligned} \tag{16}$$

Since

$$f_{y_1, \dots, y_n}^{\nu_0, \nu_1, \dots, \nu_n}(u, a_1, \dots, a_n) = 0 \quad \text{when } u > a_1 + \dots + a_n$$

and

$$f_{y_{n+1}}^{\nu_0, \nu_{n+1}}(v, a_{n+1}) = 0 \quad \text{when } v > a_{n+1}$$

provided that

$$\frac{1}{2} < \operatorname{Re}(\nu_0) < \sum_{j=1}^n \operatorname{Re}(\nu_j) + \frac{n-3}{2}, \quad \operatorname{Re}(\nu_0) - \operatorname{Re}(\nu_{n+1}) < -1, \tag{17}$$

we conclude from (16) that

$$f_{y_1, \dots, y_{n+1}}^{\nu_0, \nu_1, \dots, \nu_{n+1}}(x, a_1, \dots, a_{n+1}) = 0 \quad \text{when } x > a_1 + \dots + a_{n+1} \tag{18}$$

under (17).

The formula (18) can be analytically continued to the domain

$$-1 < \operatorname{Re}(\nu_0) < \sum_{j=1}^{n+1} \operatorname{Re}(\nu_j) + \frac{n-3}{2}.$$

Thus we have proved

Corollary. *Let*

$$-1 < \operatorname{Re}(\nu_0) < \sum_{j=1}^n \operatorname{Re}(\nu_j) + \frac{n-3}{2}, \quad a_j > 0 \quad (j = 1, \dots, n)$$

with

$$a_0 > a_1 + \dots + a_n$$

and

$$\operatorname{Re}(y_j) \geq 0 \quad (j = 1, \dots, n).$$

Then

$$\int_0^\infty t^{\nu_0+1} J_{\nu_0}(a_0 t) \prod_{j=1}^n (t^2 + y_j^2)^{-\nu_j/2} J_{\nu_j}(a_j \sqrt{t^2 + y_j^2}) dt = 0. \quad (19)$$

The formula (19) is a generalization of the formulae (2.12.44.7) (the case $y_1 = \dots = y_n = 0$) and (2.12.44.8) (the case $\operatorname{Re}(y_1) > 0, \dots, \operatorname{Re}(y_n) > 0$) in [6].

Acknowledgement. The work of the first author was supported, in part, by the National Basic Research Program in Natural Sciences, Vietnam, and by the Alexander von Humboldt Foundation.

References

- [1] F.M. Cholewinski: *Hankel Convolution Complex Inversion Theory*, Mem. Amer. Math. Soc. Vol. 58, 1965.
- [2] V.A. Ditkin and A.P. Prudnikov: *Integral Transforms and Operational Calculus*, Pergamon Press, Oxford-London-Edinburgh-New York-Paris-Frankfurt, 1965.
- [3] A. Erdélyi, W. Magnus, F. Oberhettinger and R.P. Soni: *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York-Toronto-London, 1953.
- [4] I.I. Hirschman, Jr.: Variation diminishing Hankel transforms, *J. Analyse Math.* **8**(1960/61), 307-336.
- [5] Nguyen Thanh Hai and S.B. Yakubovich: *The Double Mellin-Barnes Type Integrals and Their Applications to Convolution Theory*, World Scientific, Singapore-New Jersey-London-Hong Kong, 1992.
- [6] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev: *Integrals and Series*, Vol. 2, *Special Functions*, Gordon & Breach, New York-London-Paris-Tokyo, 1986.
- [7] J. de Sousa Pinto: A generalized Hankel convolution, *SIAM J. Math. Anal.* **16** (1985), 1335-1346.
- [8] E.C. Titchmarsh: *Introduction to the Theory of Fourier Integrals*, Oxford Univ. Press, Oxford, 1948.