ON CERTAIN CLASSES OF MEROMORPHICALLY STARLIKE FUNCTIONS

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ABSTRACT. The object of the present paper is to introduce a new class $\Sigma_n(\alpha)$ of meromorphic functions defined by a multiplier transformation and to investigate some properties for the class $\Sigma_n(\alpha)$. Further we study integrals of functions in $\Sigma_n(\alpha)$.

KEY WORDS AND PHRASES. Univalent functions, meromorphically starlike functions, integral operators.

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1. INTRODUCTION.

Let Σ denote the class of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-1} \neq 0)$$

which are regular in the punctured disk $D = \{z: 0 < |z| < 1\}$. For any integer n , let the operator I^n operating on $f \in \Sigma$ be defined by

$$I^n f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} (k+2)^{-n} a_k z^k$$

Obviously, we have

$$I^n(I^m f(z)) = I^{n+m} f(z)$$

for all integers m and n . For any nonpositive integer n , the operators I^n are the differential operators studied by Uralegaddi and Somanatha [5]. Also the operators I^n are closely related to the multiplier transformations introduced by Flett [2].

For any integer n , let $\Sigma_n(\alpha)$ denote the class of functions $f \in \Sigma$ satisfying the condition

$$\operatorname{Re} \left\{ \frac{I^{n-1}}{I^n f(z)} - 2 \right\} < -\alpha \quad (0 \leq \alpha < 1, z \in U = \{z : |z| < 1\}).$$

In this paper, we prove that for the classes $\Sigma_n(\alpha)$ of functions in Σ , $\Sigma_n(\alpha) \subset \Sigma_{n+1}(\alpha)$ holds. Since $\Sigma_0(\alpha)$ equals $\Sigma^*(\alpha)$ (the class of meromorphically starlike functions of order α), all members in $\Sigma_n(\alpha)$ are univalent for any nonpositive integer n . Further property preserving integrals are considered. Our results generalize the some results of Bajpai [1], Goel and Sohi [3] and Uralegaddi and Somanatha [6].

2. MAIN RESULTS.

We begin with the statement of the following lemma due to Miller and Mocaun [4].

LEMMA. Let $\phi(u, v)$ be a complex valued function, $\phi: D \rightarrow C, D \subset C^2 \in (C$ is the complex plane), and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies the following condition

- (i) $\phi(u, v)$ is continuous in D ,
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} > 0$;
- (iii) $\operatorname{Re}\{\phi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-(1+u_2^2)}{2}$.

Let $p(z) = 1 + p_1z + p_2z^2 + \dots$ be regular in U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$\operatorname{Re}\{\phi(p(z), zp'(z))\} > 0 \quad (z \in U),$$

then $\operatorname{Re}\{p(z)\} > 0$ ($z \in U$).

With the aid of above lemma, we drive

THEOREM 1. If $f \in \Sigma_n(\alpha)$, then $f \in \Sigma_{n+1}(\beta)$, where

$$\beta = \frac{5 + 2\alpha - \sqrt{(3 - 2\alpha)^2 + 8}}{4}. \quad (2.1)$$

PROOF. Define the function $p(z)$ by

$$\frac{I^n f(z)}{I^{n+1} f(z)} = \gamma + (1 - \gamma)p(z), \quad (2.2)$$

where

$$\gamma = \frac{(3 - 2\alpha) + \sqrt{(3 - 2\alpha)^2 + 8}}{4} \quad (\gamma > 1). \quad (2.3)$$

We see that $p(z) = 1 + p_1z + p_2z^2 + \dots$ is regular in U . Making use of the logarithmic differentiations of both sides in (2.2) and using the identity

$$z(I^n f(z))' = I^{n-1} f(z) - 2I^n f(z), \quad (2.4)$$

we obtain

$$\frac{I^{n-1} f(z)}{I^n f(z)} = \gamma + (1 - \gamma)p(z) + \frac{(1 - \gamma)zp'(z)}{\gamma + (1 - \gamma)p(z)} \quad (2.5)$$

or

$$- \operatorname{Re}\left\{\frac{I^{n-1} f(z)}{I^n f(z)} - 2 + \alpha\right\} = \operatorname{Re}\left\{2 - (\alpha + \gamma - (1 - \gamma)p(z) - \frac{(1 - \gamma)zp'(z)}{\gamma + (1 - \gamma)p(z)})\right\} > 0. \quad (2.6)$$

Let us define the function $\phi(u, v)$ by

$$\phi(u, v) = 2 - (\alpha + \gamma) - (1 - \gamma)u - \frac{(1 - \gamma)v}{\gamma + (1 - \gamma)u}. \quad (2.7)$$

Then $\phi(u, v)$ satisfies

- (i) $\phi(u, v)$ is continuous in $D = \left(C - \left\{\frac{\gamma}{\gamma - 1}\right\}\right) \times C$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = 1 - \alpha > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-(1+u_2^2)}{2}$,

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= 2 - (\alpha + \gamma) - \frac{\gamma(1-\gamma)v_1}{\gamma^2 + (1-\gamma)^2 u_2^2} \\ &\leq 2 - (\alpha + \gamma) + \frac{\gamma(1-\gamma)v_2}{2(\gamma + (1-\gamma)^2 u_2^2)} \\ &\leq 0. \end{aligned}$$

Thus the function $\phi(u, v)$ satisfies the conditions in our Lemma. This shows that $\operatorname{Re}\{p(z)\} > 0$ ($z \in U$), hence

$$\operatorname{Re}\left\{\frac{I^n f(z)}{I^{n+1} f(z)}\right\} < \gamma \quad (z \in U) \quad (2.8)$$

or

$$\operatorname{Re}\left\{\frac{I^n f(z)}{I^{n+1} f(z)} - 2\right\} < -\beta \quad (z \in U) \quad (2.9)$$

where β is given by (2.1). Therefore we complete the proof of the theorem.

Since $\beta - \alpha > 0$ in Theorem 1, we have

COROLLARY 1. $\Sigma_n(\alpha) \subset \Sigma_{n+1}(\alpha)$ for any integer n .

REMARK. For nonpositive integers n , Corollary 1 is a similar result obtained by Uralegaddi and Somanatha [6].

Putting $n = 0$ and $\alpha = 0$ in Corollary 1, we obtain the following result of Bajpai [1].

COROLLARY 2. If $f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k$ ($a_{-1} \neq 0$) is meromorphically starlike, then so is

$$F_1(z) = \frac{1}{x^2} \int_0^z t f(t) dt. \quad (2.10)$$

Next, we prove

THEOREM 2. Let $f \in \Sigma_n(\alpha)$ and let

$$F_c(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt \quad (c > 0) \quad (2.11)$$

Then $F_c \in \Sigma_n(\beta)$, where

$$\beta = \frac{(3 + 2\alpha c) - \sqrt{(5 - 2\alpha - 2c)^2 + 8(4c - 3 - 2\alpha c + 2\alpha)}}{4} \quad (2.12)$$

PROOF. Let $f \in \Sigma_n(\alpha)$. Then we have

$$\operatorname{Re}\left\{\frac{I^{n-1} f(z)}{I^n f(z) - 2}\right\} < -\alpha. \quad (2.13)$$

From the definition of F_c , we obtain

$$z(I^n F_c(z))' = c I^n f(z) - (c+1) I^n F_c(z) \quad (2.14)$$

and also

$$z(I^n F_c(z))' = I^{n-1} F_c(z) - 2I^n F_c(z). \quad (2.15)$$

Using (2.14) and (2.15), the condition (2.13) may be written as

$$\operatorname{Re} \left\{ \frac{\frac{I^{n-2} F_c(z)}{I^{n-1} F_c(z)} + (c-1)}{1 + (c-1) \frac{I^n F_c(z)}{I^{n-1} F_c(z)}} - 2 \right\} < -\alpha. \quad (2.16)$$

Define the function $p(z)$ by

$$\frac{I^{n-1} F_c(z)}{I^n F_c(z)} = \gamma + (1-\gamma)p(z), \quad (2.17)$$

where

$$\gamma = \frac{(5-2\alpha-2c) + \sqrt{(5-2\alpha-2c)^2 + 8(4c-3-2\alpha c+2\alpha)}}{4} \quad (\gamma > 1). \quad (2.18)$$

Then $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is regular in U . Differentiating (2.17) logarithmically and simplifying, we have

$$\frac{\frac{I^{n-2} F_c(z)}{I^{n-1} F_c(z)} + (c-1)}{1 + (c-1) \frac{I^n F_c(z)}{I^{n-1} F_c(z)}} - 2 = -2 + \gamma + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(\gamma+c-1) + (1-\gamma)p(z)}. \quad (2.19)$$

It follows from (2.19) that

$$\begin{aligned} & -\operatorname{Re} \left\{ \frac{\frac{I^{n-2} F_c(z)}{I^{n-1} F_c(z)} + (c-1)}{1 + (c-1) \frac{I^n F_c(z)}{I^{n-1} F_c(z)}} - 2 + \alpha \right\} \\ &= \operatorname{Re} \left\{ 2 - (\alpha + \gamma) - (1-\gamma)p(z) - \frac{(1-\gamma)zp'(z)}{(\gamma+c-1) + (1-\gamma)p(z)} \right\} \\ &> 0. \end{aligned} \quad (2.20)$$

If we define the function $\phi(u, v)$ by

$$\phi(u, v) = 2 - (\alpha + \gamma) - (1-\gamma)u - \frac{(1-\gamma)v}{(\gamma+c-1) + (1-\gamma)u}, \quad (2.21)$$

then $\phi(u, v)$ satisfies

- (i) $\phi(u, v)$ is continuous in $D = \left(C - \left\{ \frac{\gamma+c-1}{\gamma-1} \right\} \right) \times C$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = 1 - \alpha > 0$;
- (iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-(1+u_2^2)}{2}$,

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= 2 - (\alpha + \gamma) - \frac{(\gamma+c-1)(1-\gamma)v_1}{(\gamma+c-1)^2 + (1-\gamma)^2 u_2^2} \\ &\leq 2 - (\alpha + \gamma) + \frac{(\gamma+c-1)(1-\gamma)(1+u_2^2)}{2\{(\gamma+c-1)^2 + (1-\gamma)^2 u_2^2\}} \\ &\leq 0. \end{aligned}$$

Since $\phi(u, v)$ satisfies the conditions in Lemma, we have that $\operatorname{Re}\{p(z)\} > 0 (z \in U)$. This proves that

$$\operatorname{Re} \left\{ \frac{I^{n-1} F_c(z)}{I^n F_c(z)} \right\} < \gamma \quad (z \in U) \quad (2.22)$$

or

$$\operatorname{Re} \left\{ \frac{I^{n-1} F_c(z)}{I^n F_c(z)} - 2 \right\} < -\beta \quad (z \in U), \quad (2.23)$$

where β is given by (2.12). That is, $F_c(z) \in \Sigma_n(\beta)$.

Similarly, from Theorem 2, we have

COROLLARY 3. If $f \in \Sigma_n(\alpha)$, then the integral operator F_c defined by (2.11) belongs to the class $\Sigma_n(\alpha)$.

Taking $n = 0$ and $\alpha = 0$ in Corollary 3, we obtained the following corresponding result of Goel and Sohi [3].

COROLLARY 4. If $f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k$ is meromorphically starlike, then so is the integral operator F_c defined by (2.11).

The following theorem gives us a characterization of the class $\Sigma_n(\alpha)$.

THEOREM 3. $f \in \Sigma_n(\alpha)$ if and only if the integral operator F_1 defined by (2.10) belongs to the class $\Sigma_{n-1}(\alpha)$.

PROOF. For $c = 1$, the identities (2.14) and (2.15) reduce to $I^n f(z) = I^{n-1} F_1(z)$ and hence $I^{n-1} f(z) = I^{n-2} F_1(z)$. Therefore

$$\frac{I^{n-1} f(z)}{I^n f(z)} = \frac{I^{n-2} F_1(z)}{I^{n-1} F_1(z)} \quad (2.24)$$

and the result follows.

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REFERENCES

1. BAJPAI, S.K., A note on a class of meromorphic univalent functions, *Rev. Roumanie Math. Pure Appl.* **22** (1977), 295-297.
2. FLETT, T.M., The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.* **38** (1972), 746-765.
3. GOEL, R.M. and SOHI, N.S., On a class of meromorphic functions, *Glas. Mat.*, **17** (1981), 19-28.
4. MILLER, S.S. and MOCANU, P.T., Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.* **65** (1978), 289-305.
5. URALEGADDI, B.A. and SOMANATHA, C., Certain differential operators for meromorphic functions, *Houston J. Math.*, **17** (1991), 279-284.
6. URALEGADDI, B.A. and SOMANATHA, C., New criteria for meromorphic starlike univalent functions, *Bull. Aust. Math. Soc.*, **43** (1991), 137-140.