THE BEST APPROXIMATION AND AN EXTENSION OF A FIXED POINT THEOREM OF F.E. BROWDER

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ABSTRACT In this paper, the KKM principle has been used to obtain a theorem on the best approximation of a continuous function with respect to an affine map. The main result provides extensions of some well-known fixed point theorems.

KEY WORDS AND PHRASES. Best approximation, fixed point theorem, KKM-map, p-affine map, inward set.

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Let $E$ be a locally convex topological vector space and $C$ a non-empty subset of $E$. A mapping $p : C \times E \to [0, \infty)$ is a convex map iff for each fixed $x \in C$, $p(x, \cdot) : E \to (0, \infty)$ is a convex function. For $x \in C$, the inward set $I_C(x) = \{ x + r(y - x) : y \in C, r > 0 \}$. Browder [1] proved the following extension of the Schauder's fixed point theorem.

THEOREM 1. (Browder). Let $C$ be a compact, convex subset of $E$ and $f : C \to E$ a continuous map. If $p : C \times E \to [0, \infty)$ is a continuous convex map satisfying

1. for each $x \neq f(x)$, there exists a $y \in I_C(x)$ with $p(x, f(x) - y) < p(x, f(x) - x)$, then $f$ has a fixed point.

It may be stated that the importance of Theorem 1 stems from $p$ being a continuous convex map instead of a continuous seminorm on $E$. In this paper, we use the KKM principle to obtain a result on the 'best approximation' that yields Theorem 1 with relaxed hypothesis on compactness.

Let $X$ be a non-empty subset of $E$. Recall that a mapping $F : X \to 2^E$ is a KKM map if $F(x) \neq \emptyset$ for each $x \in X$, and for any finite subset $A = \{ x_1, x_2, \ldots, x_n \} \subseteq X$, $C_0(A) \subseteq \bigcup \{ F(x_i) : i = 1, 2, \ldots, n \}$, where $C_0(A)$ denotes the convex hull of $A$. Observe that if $F$ is a KKM map, then $x \in F(x)$ for each $x \in X$.

It is shown by Fan [2] that if $F : X \to 2^E$ is a closed valued KKM map, then the family $\{ F(x) : x \in X \}$ has the finite intersection property.

As an immediate consequence of the above result, we have:

LEMMA 2. If $X$ is a non-empty compact, convex subset of $E$ and $F : X \to 2^E$ is a closed valued KKM map, then $\cap \{ F(x) : x \in X \} \neq \emptyset$. 
PROOF. Define a map \( G : X \rightarrow 2^X \) by
\[
G(x) = \{ y \in X : p(y, f(y) - g(y)) \leq p(y, f(y) - g(z)) \}.
\]
Then \( G(x) \) is a nonempty compact subset of \( X \) and \( G \) is a KKM map. Consequently, by [2], \( \{ G(x) : x \in X \} \) has the finite intersection property. Since \( X \) is compact, it follows that \( \bigcap \{ G(x) : x \in X \} \neq \emptyset \), and hence, \( \bigcap \{ F(x) : x \in X \} \neq \emptyset \).\( \square \)

The following lemma is essentially due to Kim [3]. We give a proof for completeness.

Note: In the following, \( C_0(A) \) stands for the closed convex hull of \( A \).

**Lemma 3.** If \( A \) and \( B \) are compact, convex subsets of \( E \), then \( C_0(A \cup B) \) is a compact, convex subset of \( E \).

**Proof.** Since \( A \) and \( B \) are convex, it follows that \( C_0(A \cup B) = \{ \lambda x + \mu y : x \in A, y \in B, \lambda, \mu \in [0, 1] \text{ and } \lambda + \mu = 1 \} \). Clearly, \( C_0(A \cup B) \) is a closed and convex subset of \( E \). To show that \( C_0(A \cup B) \) is compact, let \( C = [0,1] \times [0,1] \times A \times B \) and \( D = \{ \lambda x + \mu y : x \in A, y \in B, \lambda, \mu \in [0,1] \} \). Then \( C \) is a compact subset of \( Y = [0,1] \times [0,1] \times E \times E \) in the product topology on \( Y \). Further, the mapping \( f : Y \rightarrow E \) defined by \( f(\lambda, \mu, x, y) = \lambda x + \mu y \) being continuous, it follows that \( D = f(C) \) is a compact subset of \( E \) and, hence, \( C_0(A \cup B) \subseteq D \) is compact. \( \square \)

**Lemma 4.** Let \( X \) be a non-empty convex subset of \( E \) and \( F : X \rightarrow 2^E \) a closed valued KKM map. If there exists a compact, convex set \( S \subseteq E \) such that \( \cap \{ F(x) : x \in S \} \) is non-empty and compact, then \( \cap \{ F(x) : x \in X \} \neq \emptyset \).

**Proof.** Let \( C = \cap \{ F(x) : x \in S \} \). Then \( C \) is non-empty and a compact subset of \( E \). To prove the lemma, it suffices to show that the family \( \{ F(x) \cap C : x \in X \} \) has the finite intersection property. To prove this, let \( A \) be a finite subset of \( X \). Then \( C_0(A) \) is compact and by Lemma 3, \( D = C_0(S \cup C_0(A)) \) is a compact convex subset of \( X \). Consequently, by Lemma 2, \( \cap \{ F(x) : x \in D \} \neq \emptyset \). This implies that \( \cap \{ F(x) \cap C : x \in A \} \neq \emptyset \). Thus, \( \{ F(x) \cap C : x \in X \} \) has the finite intersection property. Since \( C \) is compact and \( F(x) \) is closed for each \( x \in X \), it follows that \( \cap \{ F(x) \cap C : x \in X \} \neq \emptyset \). This implies that \( \cap \{ F(x) : x \in X \} \neq \emptyset \).

Let \( X \) be a non-empty convex subset of \( E \) and \( p : X \times E \rightarrow [0, \infty) \) a convex map. A mapping \( g : X \rightarrow X \) is a \( p \)-affine map iff for each triple \( \{ x, x_1, x_2 \} \subseteq X, y \in E, \) and \( \lambda, \mu \in [0,1] \) with \( \lambda + \mu = 1 \),
\[
p(x, y - g(\lambda x_1 + \mu x_2)) \leq \max \{ p(x, y - g(x_i)) : i = 1, 2 \}.
\]
**Note:** If \( g \) is linear or affine in the sense of Prolla [4], then \( p \) being convex, it follows that \( g \) is \( p \)-affine in the above sense. It is immediate that if \( g \) is \( p \)-affine, then for any finite set \( A = \{ x_1, x_2, \ldots, x_n \} \subseteq X \) and \( \lambda_i \geq 0 \) with \( \sum_{i=1}^{n} \lambda_i = 1 \),
\[
p(x, y - g(\sum_{i=1}^{n} \lambda_i x_i)) \leq \max \{ p(x, y - g(x_i)) : i = 1, 2, \ldots, n \}
\]
for each \( x \in X, y \in E. \)\( \square \)

The following is the main result of this paper.

**Theorem 5.** Let \( X \) be a nonempty convex subset of \( E \) and \( p : X \times E \rightarrow [0, \infty) \) a continuous convex map. Let \( f : X \rightarrow E \) and \( g : X \rightarrow X \) be continuous mappings with \( g \) \( p \)-affine. Suppose there exist a compact, convex set \( S \subseteq X \) and a compact set \( K \subseteq X \) such that

(2) for each \( x \in X \setminus K \) there exists an \( x \in S \) such that \( p(y, f(y) - g(y)) > p(y, f(y) - g(x)) \). Then there exists a \( u \in X \) that satisfies

(3) \( p(u, f(u) - g(u)) = \inf \{ p(u, f(u) - g(x)) : x \in X \} = \inf \{ p(u, f(y) - z) : z \in \text{cl} \ I_X(g(u)) \}. \)

**Proof.** We first prove the left equality. For this, we define a mapping \( G : X \rightarrow 2^X \) by
\[
G(x) = \{ y \in X : p(y, f(y) - g(y)) \leq p(y, f(y) - g(x)) \}.
\]
Clearly, \( x \in G(x) \) and it follows that \( G(x) \) is closed for each \( x \in X \). We show that \( G \) is a KKM map.

Let \( y = \sum_{i=1}^{n} \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, x_i \in X \) for each \( i \). Suppose \( y \notin \bigcup \{G(x_i), i = 1, 2, \ldots, n\} \).

Then for each \( i = 1, 2, \ldots, n, \)
\[
p(y, f(y) - g(y)) > p(y, f(y) - g(x_i)).
\]

This implies that \( p(y, f(y) - g(y)) = p(y, f(y) - g(\sum_{i=1}^{n} \lambda_i x_i)) \leq \max \{p(y, f(y) - g(x_i)), i = 1, 2, \ldots, n\} < p(y, f(y) - g(y)). \) This inequality is impossible and, consequently, \( y \notin \bigcup \{G(x_i) : i = 1, 2, \ldots, n\} \), that is, \( G \) is a closed valued map. Now, since \( S \) is a compact convex subset of \( X \), it follows by Lemma 2 that \( C = \cap \{G(x) : x \in S\} \) is a nonempty closed subset of \( X \). We show that \( C \subseteq K \). Suppose \( y \in C \) and assume that \( y \notin X \setminus K \). Then by hypothesis there exists an \( x \in S \) such that \( p(y, f(y) - g(y)) > p(y, f(y) - g(x)) \). This implies that \( y \notin G(x) \) for \( x \in S \) and, hence, \( y \notin C \), contradicting the initial supposition. Thus, \( C \subseteq K \) and, hence, \( \cap \{G(x) : x \in S\} \) is a nonempty compact subset of \( K \). Hence by Lemma 4, \( \cap \{G(x) : x \in X\} \neq \phi \).

If \( u \in \cap \{G(x) : x \in X\} \), then for each \( x \in X \), \( p(u, f(u) - g(u)) \leq p(u, f(u) - g(x)) \). Further, since \( u \in X \), it follows that \( p(u, f(u) - g(u)) = \inf \{p(u, f(u) - g(x)) : x \in X\} \).

This proves the first equality in (3). To prove right side of the equality in (3) we first show that for each \( z \in I_X(g(u)) \setminus X \), \( p(u, f(u) - g(u)) \leq p(u, f(u) - z) \). Now \( z \in I_X(g(u)) \setminus X \) implies that there is a \( y \in X \) and \( r > 1 \) such that \( y = \frac{1}{r} z + (1 - \frac{1}{r}) g(u) \). Hence, by the first equality and \( p \) being convex, it follows that \( p(u, f(u) - g(u)) \leq \frac{1}{r} p(u, f(u) - y) \leq \frac{1}{r} p(u, f(u) - z) + (1 - \frac{1}{r}) p(u, f(u) - g(u)) \).

That is, \( p(u, f(u) - g(u)) \leq p(u, f(u) - z) \) for each \( z \in I_X(g(u)) \setminus X \). Since the last inequality is also true for any \( z \in X \), it follows that \( p(u, f(u) - g(u)) \leq p(u, f(u) - z) \) for each \( z \in I_X(g(u)) \).

Further, since the functions \( f, g, \) and \( p \) are continuous and \( g(u) \in I_X(g(u)) \), it follows that \( p(u, f(u) - g(u)) = \inf \{p(u, f(u) - z) : z \in \text{cl} (I_X(g(u)))\} \). This proves the second equality in (3).

As a simple consequence of Theorem 2, we have

**Corollary 6.** Suppose \( X \) is a compact, convex subset of \( E, p : X \times E \to [0, \infty) \) a continuous convex function and \( f : X \to E \) a continuous function. Then for any continuous \( p \)-affine map \( g : X \to X \), there exists a \( u \in X \) that satisfies (3). Further,

(i) if \( f(x) \in \text{cl} (I_X(g(x))) \) for each \( x \in X \) then \( p(u, f(u) - g(u)) = 0, \)

(ii) if for each \( x \in X, \) with \( f(x) \neq g(x) \) there exists a \( y \in \text{cl} (I_X(g(x))) \) such that \( p(x, f(x) - g(x)) \), then \( f(u) = g(u) \).

**Proof.** Set \( S = K = X \) in Theorem 5. Since \( X \setminus K = \phi \), condition (2) in Theorem 5 is satisfied. Hence, there is a \( u \in X \) that satisfies (3). Clearly, (i) implies \( p(u, f(u) - g(u)) = 0 \). To prove (ii), suppose \( f(u) \neq g(u) \). Then by hypothesis \( p(u, f(u) - z) < p(u, f(u) - g(u)) \) for some \( z \in \text{cl} (I_X(g(u))) \). The last inequality contradicts (3). Hence, \( f(u) = g(u) \).

It may be remarked that if \( g \) is the identity mapping of \( X \), then Corollary 6 yields Browder’s Theorem 1 and also extends a recent result of Sehgal, Singh, and Gaste [5] if \( f \) therein is a single valued map.

For the next result, let \( P \) denote the family of nonnegative continuous convex functions on \( X \times E \). Note if \( p_1 \) and \( p_2 \) in \( P \), then so is \( p_1 + p_2 \). Also, if \( p \) is a continuous seminorm on \( E \), then \( p \) generates a nonnegative continuous convex function on \( X \times E \) defined by \( p(x, y) = p(y) \). A mapping \( g : X \to X \) is \( P \) affine if it is \( p \)-affine for each \( p \in P \).

The result below is an extension of an earlier result of Fan.

**Theorem 7.** Let \( X \) be a compact, convex subset of \( E \) and \( f : X \to E \) a continuous function. Then for any continuous \( P \)-affine map \( g : X \to X \),

(4) either \( f(u) = g(u) \) for some \( u \in X \),
(5) or there exists a \( p \in P \) and a \( u \in X \) with \( 0 < p(u, f(u) - g(u)) = \inf \{ p(u, f(u) - z) : z \in \text{cl}(I_X(g(u))) \} \).

In particular, if \( f(x) \in \text{cl}(I_X(g(x))) \) for each \( x \), then (5) holds.

PROOF. It follows by Theorem 5 that for each \( p \in P \) there is a \( u_p \in X \) such that
\[
p(u, f(u) - g(u)) = \inf \{ p(u, f(u) - z) : z \in \text{cl}(I_X(g(u))) \}.
\]
If for some \( p \), \( p(u, f(u) - g(u)) > 0 \), then (5) is true. Suppose then, \( p(u, f(u) - g(u)) = 0 \) for each \( p \in P \). Set \( A_p = \{ u \in X : p(u, f(u) - g(u)) = 0 \} \). Then \( A_p \) is a nonempty compact subset of \( X \). Furthermore, the family \( \{ A_p : p \in P \} \) has the finite intersection property. Consequently, there is a \( u \in X \) that satisfies
\[
(6) \quad p(u, f(u) - g(u)) = 0 \text{ for each } p \in P.
\]
If \( f(u) \neq g(u) \), then since \( E \) is separated, there exists a continuous seminorm \( p \) on \( E \) such that
\[
p(f(u) - g(u)) \neq 0 \text{ and, hence, } p(u, f(u) - g(u)) > 0, \text{ contradicting (6). Thus, } f(u) = g(u). \text{ Hence, (5) holds in the alternate case. \( \square \)

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