A CHARACTERIZATION OF THE ROGERS q-HERMITE POLYNOMIALS

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ABSTRACT. In this paper we characterize the Rogers q-Hermite polynomials as the only orthogonal polynomial set which is also $\mathcal{D}_q$-Appell where $\mathcal{D}_q$ is the Askey-Wilson finite difference operator.

KEY WORDS AND PHRASES. Orthogonal polynomials, generating functions, Askey-Wilson operator

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1. INTRODUCTION

Appell polynomials sets \{P_n(x)\} are generated by the relation

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

where $A(t)$ is a formal power series in $t$ with $A(0) = 1$. This definition implies the equivalent property that

$$D P_n(x) = P_{n-1}(x), \quad D = d/dx,$$

Examples of such polynomial sets are

$$\left\{ \frac{x^n}{n!} \right\}, \left\{ \frac{B_n(x)}{n!} \right\}, \left\{ \frac{H_n(x)}{2^n n!} \right\}$$

where $B_n(x)$ is the nth Bernoulli polynomial and $H_n(x)$ is the nth Hermite polynomials generated by

$$e^{2xt} - 1 = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$ 

By an orthogonal polynomial set (OPS) we shall mean those polynomial sets which satisfy a three term recurrence relation of the form

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad (n = 0, 1, 2, \cdots)$$

with $P_0(x) = 1, \ P_{-1}(x) = 0$, and $A_n A_{n-1} C_n > 0$.

By Favard's theorem [7] this is equivalent to the existence of a positive measure $d\alpha(x)$ such that

$$\int_{-\infty}^{\infty} P_n(x) P_m(x) \, d\alpha(x) = K_n \delta_{nm}.$$
As we see from the examples (1.3) some Appell polynomials are orthogonal and some are not. This prompted Angelesco [3] to prove that the only orthogonal polynomial sets which are also Appell is the Hermite polynomial set. This theorem was rediscovered by several authors later on (see, e.g., [10]).

There were several extensions and/or analogs of Appell polynomials that were introduced later. Some are based on changing the operator \( D \) in (1.2) into another differentiation-like operator or by replacing the generating relation (1.1) by a more general one. In most of these cases theorems like Angelesco's were given. For example Carlitz [6] proved that the Charlier polynomials are the only OPS which satisfy the difference relation

\[
\Delta P_n(x) = P_{n-1}(x), \quad (\Delta f(x) = f(x + 1) - f(x)).
\]

See [1] for many other references.

A new and very interesting analog of Appell polynomials were introduced recently, as a byproduct of other considerations, by Ismail and Zhang [9]. In discussing the Askey-Wilson operator they defined a new q-analog of the exponential function \( e^x \). This we describe in the next section.

2. NOTATIONS AND DEFINITIONS

The Askey-Wilson operator is defined by

\[ D_q f(x) = \frac{\delta_q f(x)}{\delta_q x}, \]

where \( x = \cos \theta \) and

\[ \delta_q g(e^{i\theta}) = g(q^{1/2}e^{i\theta}) - g(q^{-1/2}e^{i\theta}). \]

We further assume that \(-1 < q < 1\) and use the notation

\[
(a; q)_0 = 1, \quad (a; q)_n = (1 - a)(1 - qa) \cdots (1 - aq^{n-1}), \quad (n = 1, 2, \ldots) \tag{2.3}
\]

\[
(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k). \tag{2.4}
\]

There are two q-analogs of the exponential function \( e^x \) given by the infinite products

\[
e_q(x) = \frac{1}{(x; q)_{\infty}} = \sum_{k=0}^{\infty} \frac{x^k}{(q; q)_k}, \tag{2.5}
\]

and

\[
\frac{1}{e_q(x)} = (x; q)_{\infty} = \sum_{k=0}^{\infty} (-1)^k \frac{q^{1/2(k-1)}}{(q; q)_k} x^k. \tag{2.6}
\]

We shall also use the function

\[
\Psi_n(x) = i^n (i q^{(1-n)/2} e^{i\theta}; q)_n (i q^{(1-n)/2} e^{-i\theta}; q)_n, \tag{2.7}
\]

so that

\[
\Psi_{2n}(x) = \prod_{k=0}^{n-1} \left[ 4x^2 + (1 - q^{2n-1-2k})(1 - q^{1-2n+2k}) \right]
\]

\[
\Psi_{2n+1}(x) = 2x \prod_{k=0}^{n-1} \left[ 4x^2 - (1 - q^{2n-2k})(1 - q^{2n+2k}) \right]
\]

\[
4x^3 \Psi_n(x) = \Psi_{n+2}(x) + (1 - q^{n+1})(1 - q^{-n-1}) \Psi_n(x) \tag{2.8}
\]
Thus
\[ D_q \Psi_n(x) = 2q^{(1-n)/2} \frac{1 - q^n}{1 - q} \Psi_{n-1}(x). \] (2.9)

and
\[ D_q [x \Psi_n(x)] = \frac{q^{(n+1)/2} - q^{-(n+1)/2}}{q^{1/2} - q^{-1/2}} 2x \Psi_{n-1}(x). \] (2.10)

Iterating (2.9) we get
\[ D_q^k \Psi_n(x) = 2^k q^{k(k+1)/2 - nk} \frac{(q; q)_n}{(q; q)_{n-k}(1 - q)^k} \Psi_{n-k}(x). \] (2.11)

The Ismail-Zhang q-analog of the exponential function [9] is
\[ E(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}(1 - q)^n}{2^n(q; q)_n} \Psi_n(x) t^n. \] (2.12)

It follows from (2.12) and (2.9) that
\[ D_q E(x) = t E(x). \] (2.13)

This suggested to Ismail and Zhang to define the $D_q$-Appell polynomials as those, in analogy with (1.1), defined by
\[ A(t) E(x) = \sum_{n=0}^{\infty} P_n(x) t^n, \] (2.14)

so that
\[ D_q P_n(x) = P_{n-1}(x). \] (2.15)

An example of such a set is the Rogers q-Hermite polynomials, \{H_n(x|q)\}, (see [2, 4, 8]).

They satisfy the three term recurrence relation
\[ H_{n+1}(x|q) = 2x H_n(x|q) - (1 - q^n) H_{n-1}(x|q), \quad n = 0, 1, 2, 3, ... \] (2.17)

with $H_0(x|q) = 1$, $H_{-1}(x|q) = 0$.

3. THE MAIN RESULT

We now state our main result:

Theorem 1. The orthogonal polynomial sets which are also $D_q$-Appell, i.e., satisfy (2.15) or (2.14) is the set of the Rogers q-Hermite polynomials.

Proof Let \{Q_n(x)\} be a polynomial set which is both orthogonal and $D_q$-Appell. That is \{Q_n(x)\} satisfy (2.14) and (1.5).

We next note that (2.16) implies that
\[ h_n(x|q) = \frac{(1 - q)^n q^{n(n-1)/4}}{2^n(q; q)_n} H_n(x|q) \] (3.1)

satisfy
\[ D_q h_n(x|q) = h_{n-1}(x|q), \] (3.2)
so that \( \{h_n(x|q)\} \) is a \( D_q \)-Appell polynomial set and at the same time is an OPS satisfying the three term recurrence relation

\[
(1 - q^{n+1})h_{n+1}(x|q) = (1 - q)q^{n/2}x h_n(x|q) - \frac{1}{4}(1 - q)^2 q^{-n/2}h_{n-1}(x|q)
\]  

(3.3)

It also follows from (2.14) that any two polynomial sets \( \{R_n(x)\} \) and \( \{S_n(x)\} \), in that class are related by \( R_n(x) = \sum_{k=0}^n c_{n-k} S_k(x) \). Thus the solution to our problem may be expressed as

\[
Q_n(x) = \sum_{k=0}^n a_{n-k} h_k(x|q).
\]  

(3.4)

for some sequence of real constants \( \{a_n\} \). We may assume without loss of generality that \( a_0 = 1 \).

The three term recurrence relation satisfied by \( \{Q_n(x)\} \) is

\[
(1 - q^{n+1})Q_{n+1}(x) = \left( (1 - q)q^{n/2}x + \beta_n \right) Q_n(x) - \gamma_n Q_{n-1}(x),
\]  

(3.5)

with \( Q_0(x) = 1 \), \( Q_{-1}(x) = 0 \). Thus \( Q_1(x) = x + \beta_0 = a_1 + h_1(x|q) \), from which it follows that \( a_1 = \beta_0 \).

Putting (3.4) in (3.5) and using (3.3) to replace \( x h_k(x|q) \) in terms of \( h_{k+1}(x|q) \) and \( h_{k-1}(x|q) \) we get, on equating coefficients of \( h_k(x|q) \),

\[
(1 - q^{n-k+1/2})(1 + q^{(n+k)/2})a_{n+1-k} - \beta_n a_{n-k} + \left[ \gamma_n - \frac{1}{4}(1 - q)^2 q^{(n+k)/2} \right] a_{n-k-1} = 0,
\]  

valid for all \( n \) and \( k = 0, 1, 2, ..., n+1 \) provided we interpret \( a_{-1} = a_{-2} = 0 \). It is easy to see that this system of equations is equivalent to the solution of our problem.

Putting \( k = n \) in (3.6) we get

\[
\beta_n = (1 - q^{1/2})(1 + q^{n+1/2})a_1.
\]  

(3.7)

Hence if \( \beta_0 = 0 \) then \( \beta_n = 0 \) for all \( n \). In fact if \( \beta_m = 0 \) for any \( n = m \) then \( \beta_n = 0 \) for all \( n \).

Now we treat these two cases separately.

Case I. \( (\beta_0 = 0) \).

The system (3.6) can now be written as

\[
(1 - q^{k+1/2})(1 + q^{k+1/2})a_{k+1} + \left[ \gamma_n - \frac{1}{4}(1 - q)^2 q^{k+1} \right] a_{k-1} = 0.
\]  

(3.8)

Since \( a_1 = 0 \) then it follows from (3.8) that \( a_{2k+1} = 0 \) for all \( k \). In particular we get

\[
\gamma_n = \frac{1}{4}(1 - q)^2 q^{n-1/2} - a_2(1 - q)(1 + q^n),
\]  

(3.9)

so that if \( a_2 = 0 \) then

\[
Q_n(x) = h_n(x|q).
\]  

(3.10)

Now we show that \( a_2 \neq 0 \) leads to contradiction. To do this replace \( k \) by \( 2k - 1 \). We get

\[
(1 - q^k)(1 + q^{n-k+1})a_{2k} + \left[ \frac{1}{4}(1 - q)^2 q^{n-1/2}(1 - q^{1-k}) - a_2(1 - q)(1 + q^n) \right] a_{2k-2} = 0.
\]  

(3.11)

Keep \( k \) fixed and let \( n \to \infty \). We get \( (1 - q^k)a_{2k} = (1 - q)a_2 a_{2k-2} \). Thus

\[
a_{2k} = \frac{(1 - q)^k}{(q;q)_k} a_2.
\]  

(3.12)

Putting this value in (3.11) we get \( q^{1-k} = 1 \). This is a contradiction and Case I is finished.

Case II \( (\beta_0 \neq 0) \).
We start with (3.6) we get, assuming $a_1 \neq 0$,
\[
\gamma_n = \frac{1}{4}(1 - q)^2 q^{n-\frac{1}{4}} + (1 - q^\frac{1}{2})(1 + q^{n+\frac{1}{2}})a_1^2 - (1 - q)(1 + q^n)a_2.
\] (3.13)

Putting this value of $\gamma_n$ and the value of $\beta_n$ in (3.7) in (3.6), and finally equating coefficients of $q^n$ and the terms independent of $n$ we get the pair of equation systems
\[
(1 - q^{(k+1)/2})a_{k+1} - (1 - q^\frac{1}{2})a_1a_k + \left\{(1 - q^\frac{1}{2})a_1^2 - (1 - q)a_2\right\} a_{k-1} = 0
\] (3.14)
and
\[
\left\{\frac{1}{4}(1 - q)^2 q^{-\frac{1}{2}}(q^{(k-1)/2} - 1) + q^{k/2}(1 - q^\frac{1}{2})a_1^2 - (1 - q)q^{(k-1)/2}a_2\right\} a_{k-1} = 0
\] (3.15)

Eliminating $a_{k+1}$ in these equations we get
\[
(1 - q^\frac{1}{2})(1 - q^{k/2})a_1a_k + \left\{(1 - q)a_2(1 - q^{(k-1)/2})\right\} (1 - q^\frac{1}{2})a_{k-1} = 0
\] (3.16)
This equation is of the form $(1 - q^{k/2})a_1a_k = c(1 - bq^{k/2})a_{k-1}$ so that the general solution of (3.16) is
\[
a_k = c_k^q \frac{(q^{\frac{1}{2}}; q^\frac{1}{2})_k}{(q^\frac{1}{2}; q^\frac{1}{2})_k}
\] (3.17)

Putting this in (3.14) we get that $b = 0$. On the other hand (3.15) gives that $c^2 = \frac{1}{4}(1 - q)^2 q^{-\frac{1}{2}}$. Finally putting those values of $a_k$ in (3.13) we get that $\gamma_n = 0$ which is a contradiction.

This completes the proof of the theorem.

### 4. GENERATING FUNCTION

We obtain, for the q-Hermite polynomials, a generating function of the form (2.14). More specifically we prove

**Theorem 2.** Let $H_n(x|q)$ be the $n$th Rogers q-Hermite polynomial. Then we have
\[
\sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{(q; q)_n} H_n(x|q) t^n = (t^2 q^{1/2}; q^2)_\infty \mathcal{E}(x).
\] (4.1)

**Proof.** Let $A(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ and
\[
A(t)\mathcal{E}(x) = \sum_{n=0}^{\infty} h_n(x|q)t^n.
\] (4.2)

Then we get
\[
h_n(x|q) = \sum_{k=0}^{n} a_{n-k} c_k \Psi_k(x).
\] (4.3)
where
\[
c_k = \frac{(1 - q)^k}{2^k (q; q)_k} q^{(k-1)/4}.
\] (4.4)
To calculate the coefficients \( \{a_n\} \) we first iterate (3.3) we get

\[
4x^2 h_n(x|q) = \frac{4}{(1-q)^2}(1-q^{n+1})(1-q^{n+2})q^{-n-\frac{3}{2}}h_{n+2}(x|q) + (2-q^n-q^{n+1})h_n(x|q) + \frac{(1-q)^2}{4}q^{-\frac{3}{2}}h_{n-2}(x|q).
\] (4.5)

Putting (4.3) in (4.5), using (2.6) and then equating coefficients of \( \Psi_k(x) \) we get after some simplification

\[
4(1-q^2)q^{-n-\frac{1}{2}}(1-q^{n-k+2})(1-q^{n+k+1})a_{n+2-k} + q^{-k-1}\left\{1 + q^{2k+2} - q^{n+k+1} - q^{n+k+2}\right\}a_{n-k} + \frac{(1-q)^2}{4}q^{-\frac{3}{2}}a_{n-2-k} = 0 \quad (k = 0, 1, \ldots, n + 2).
\] (4.6)

By direct calculation of \( a_1, a_2, a_3 \) we see easily that \( a_1 = a_3 = 0 \). Thus (4.6) shows that \( a_{2k+1} = 0 \) for all \( k \).

Furthermore we can easily verify that

\[
a_{2j} = (-1)^j \frac{(1-q)^{2j}}{2^{2j}(q^2; q^2)^j}, j = 0, 1, 2, 3, \ldots
\] (4.7)

Hence

\[
A(t) = \sum_{j=0}^{\infty} (-1)^j \frac{q^{j(j-1)}}{(q^2; q^2)^j} \left( \frac{(1-q)^2}{4} q^{-\frac{3}{2}} \right)^j
\] (4.8)

After some rescaling we get the theorem.

As a corollary of (4.1) we state the pair of inverse relations

\[
\Psi_n(x) = \sum_k \frac{(q; q)_n q^{k(k-n)}}{(q^2; q^2)_k (q; q)_{n-2k}} H_{n-2k}(x|q),
\] (4.9)

\[
H_n(x|q) = \sum_k (-1)^k \frac{(q; q)_n q^{(2k-n-1)}}{(q^2; q^2)_k (q; q)_{n-2k}} \Psi_{n-2k}(x).
\] (4.10)

These follows from the identities (2.5) and (2.6).

Formula (4.10) and (2.11) give

\[
H_n(x|q) = \frac{1}{e_{q^2}(\frac{1-q^2}{4} q^{-\frac{1}{2}}D_x^2)} \Psi_n(x).
\] (4.11)

This is a \( q \)-analog of the formula

\[
e^{-D_x^2} x^n = H_n(x)
\]

for the regular Hermite polynomials (1.4).

References


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