ON CERTAIN SEQUENCE SPACES II

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ABSTRACT In this paper we define the space $c_0(\Delta) = \{x = (x_k) / x_k - x_{k-1} \rightarrow 0 \text{ (} k \rightarrow \infty \text{)}, x_0 = 0, x_k \in \mathbb{C}\}$ and compute its duals (Continuous dual, β -dual and N-dual) The aim of this paper is to give same results about matrix mapping of $c_0(\Delta)$ into other sequence spaces including the convergent sequences, null sequences and bounded sequences

KEY WORDS AND PHRASES: Sequence spaces, matrix maps, Δ -norm, β -dual, Null-dual

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1. Introduction

Let l_∞ , c and c_0 be the linear spaces of complex bounded, convergent and null sequences $x = (x_k)$ respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in \mathbb{N} = \{1, 2, \dots\}$ the positive integers. On the other hand we defined

$$l_\infty(\Delta) = \{x = (x_k) / \Delta x \in l_\infty\}, c(\Delta) = \{x = (x_k) / \Delta x \in c\} \text{ and } c_0(\Delta) = \{x = (x_k) / \Delta x \in c_0\} \text{ where } \Delta x = (x_k - x_{k-1}), x_0 = 0 \text{ [2].}$$

(Throughout this paper it is assumed that $x_0 = 0$)

$c_0(\Delta)$, $c(\Delta)$ and $l_\infty(\Delta)$ are Banach Spaces with the norm

$$\|x\|_\Delta = \sup_k |x_k - x_{k-1}| \quad [2].$$

c_0 , c , l_∞ and $M_0 = l_\infty \cap c_0(\Delta)$ are Banach with the norm $\|\cdot\|_\infty$ but they aren't Banach with the norm $\|\cdot\|_\Delta$.

If we say $sx = (\sum_{k=1}^n x_k)$ then we have $m_s = \{x = (x_k) / sx \in l_\infty\}$, $c_s = \{x = (x_k) / sx \in c\}$ and $(c_0)_s = \{x = (x_k) / sx \in c_0\}$

[4]. l_∞ , c and c_0 are isometrically isomorphic to m_s , c_s and $(c_0)_s$, respectively with their natural norms.

For instance $f: l_\infty \rightarrow m_s$, $f(x) = \Delta x$ and $f^{-1}: m_s \rightarrow l_\infty$

$f^{-1}(x) = sx$ are isometric isomorphisms. Similarly $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are isometrically isomorphic to l_∞ ,

c and c_0 respectively. Obviously

$$f: (c_0(\Delta), \|\cdot\|_\Delta) \rightarrow (c_0, \|\cdot\|_\infty), f(x) = \Delta x$$

and

$$f^{-1}: (c_0, \|\cdot\|_\infty) \rightarrow (c_0(\Delta), \|\cdot\|_\Delta), f^{-1}(x) = sx$$

are isometric isomorphisms.

(1.1)

We have investigated matrix maps and related questions connected with $l_\infty(\Delta)$ and $c(\Delta)$ in [2]. We know that c_0 and c have Schauder basis but l_∞ has no basis with the norm $\|\cdot\|_\infty$. Write $e_k = (0, 0, \dots, 0, \underset{k}{1}, 0, \dots)$. Then (e_k) is a basis for c_0 and (e_{k-1}) ($e_0 = (1, 1, 1, \dots)$) is a basis for c , with $\|\cdot\|_\infty$ and $\|\cdot\|_\Delta$. On the other hand $(E_k) = (0, 0, \dots, 0, \underset{k-1}{1}, 1, 1, \dots)$ is a basis for M_0 and $c_0(\Delta)$ with the norm $\|\cdot\|_\Delta$. So $c_0(\Delta)$ is a separable Banach Space.

We know that the continuous dual of c_0 and c is $l_1 = \{x = (x_k) / \sum_{k=1}^\infty |x_k| < \infty, x_k \in \mathbb{C}\}$ [3] (Page 110) ($\mathbb{C}$ the set of complex numbers) Thus l_1 is continuous dual of $c_0(\Delta)$ by (1.1) Moreover, we can prove that

$$\bar{c} = \bar{M}_0 = c_0(\Delta)$$

with the norm $\|\cdot\|_{\Delta}$, where the bar denotes closure. For this, let $x \in c_0(\Delta)$ and $\epsilon > 0$ be any number. Then there exists one and only one $y = (y_k) \in c_0$ such that $x_k = \sum_{i=1}^k y_i$ (1.1) and a corresponding index $M = M(\epsilon) \in \mathbb{N}$ such that $|y_k| < \epsilon/2$ for all $k \geq M$. Now we take

$$z_k = \begin{cases} x_k, & 1 \leq k \leq M \\ x_M, & k > M \end{cases},$$

thus $z = (z_k) \in c \subset c_0(\Delta)$ belongs to the open ball $B(x, \epsilon)$ which is in $(c_0(\Delta), \|\cdot\|_{\Delta})$.

2. β -dual, N-dual and Matrix Maps

If X is a sequence space, we define

$X^{\beta} = \{a = (a_k) / \sum_{k=1}^{\infty} a_k x_k \text{ is Convergent for each } x \in X\}$

$X^N = \{a = (a_k) / \lim_k a_k x_k = 0, \text{ for each } x \in X\}$. X^{β} is called the β -(or generalized Köthe-Toeplitz) dual [1] and we will say that X^N is N-(or null) dual space of X . We have that if $X \subset Y$, then $Y^{\beta} \subset X^{\beta}$. The N-dual has similar properties with the β -dual. For instance if $X \subset Y$ then $Y^N \subset X^N$ and $X^{\beta} \subset X^N$.

$$\text{Obviously } c_0^N = l_{\infty}^N, l_{\infty}^N = M_0^N = c^N = c_0.$$

$c^N(\Delta) = l_{\infty}^N(\Delta) = \{a = (a_k) / (ka_k) \in c_0\}$. Let (X, Y) denote the set of all infinite matrices $A = (a_{nk})$ which map X into Y .

LEMMA 1. Let $(a_k) \in l_1$ and if $\lim_k |a_k x_k| = L$ exists for an $x \in c_0(\Delta)$, then $L = 0$.

Proof. It is trivial if $x = (x_k)$ is bounded. Suppose that $x \in c_0(\Delta)$ is unbounded and $\lim_k |a_k x_k| = L > 0$.

Then x can't have a bounded subsequence. If (x_{k_n}) is bounded then $\lim_n |a_{k_n} x_{k_n}| = 0$ implies $L = 0$. So

we can take $x_{k_n} \neq 0$ for all $n \in \mathbb{N}$.

Now let $\epsilon = \frac{L}{2} > 0$, then there exists an $M_1 = M_1(\epsilon) \in \mathbb{N}$ such that $\frac{L}{2} < |a_k x_k| < \frac{3L}{2}$ for all $k \geq M_1$. Thus we get $|a_k| > \frac{L}{2} \frac{1}{|x_k|}$ for all $k \geq M_1$ and

$$\sum_{k=1}^{\infty} \frac{1}{|x_k|} < \infty \quad (2.1)$$

We have that $\frac{x_k}{k} \rightarrow 0$ ($k \rightarrow \infty$) [2]. Let $\epsilon = 1$, then we have $\frac{|x_k|}{k} < 1$ and $\frac{1}{|x_k|} > \frac{1}{k}$ for all $k \geq M_2(1) \in \mathbb{N}$. If we

take $\max\{M_1, M_2\} = M$ then $\sum_{k=1}^{\infty} \frac{1}{|x_k|} \geq \sum_{k=M}^{\infty} \frac{1}{|x_k|} = \infty$. This contradicts with (2.1). So L must be zero.

LEMMA 2. $c_0^N(\Delta) = \{a = (a_k) / (ka_k) \in l_{\infty}\} = E$.

Proof. Suppose that $a = (a_k) \in E$. Since $\lim_k \frac{x_k}{k} = 0$ for all $x = (x_k) \in c_0(\Delta)$ [2], then we get

$\lim_k a_k x_k = \lim_k ka_k \frac{x_k}{k} = 0$. This implies that $a \in c_0^N(\Delta)$.

Now let $a \in c_0^N(\Delta)$. Then $\lim_k a_k x_k = 0$, for all $x \in c_0(\Delta)$, then there exists one and only one $y = (y_k) \in c_0$.

such that $x_n = \sum_{k=1}^n y_k$ (1.1)

$\lim_n a_n x_n = \lim_n \sum_{k=1}^n a_n y_k = 0$ for all $y = (y_k) \in c_0$. If we take

$$a_{nk} = \begin{cases} a_n, & 1 \leq k \leq n \\ 0, & k > n \end{cases}$$

we get $\lim_n \sum_{k=1}^{\infty} a_{nk} y_k = 0$, for all $x \in c_0$. Then $A = (a_{nk}) \in (c_0, c_0)$ and we have

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| = \sup_n \sum_{k=1}^n |a_n| = \sup_n n |a_n| < \infty \quad [4] \quad \text{This completes the proof}$$

For the next results we introduce the sequence (R_k) [resp. matrix R] given by $R_k = \sum_{i=k}^{\infty} a_i$ [resp. matrix $R = (R_{nk}) = (\sum_{i=k}^{\infty} a_{ni})$].

LEMMA 3. $c_0^{\beta}(\Delta) = \{a = (a_k) \in I_1 / (R_k) \in I_1 \cap c_0^N(\Delta)\} = D$

Proof. Suppose that $a \in D$. If $x \in c_0(\Delta)$ then we use Abel's summation formula to get

$$\begin{aligned} \sum_{k=1}^n a_k x_k &= \sum_{k=1}^n \left(\sum_{i=1}^k a_i \right) (x_k - x_{k+1}) + \left(\sum_{k=1}^n a_k \right) x_{n+1} \\ &= \sum_{k=1}^n (R_1 - R_{k+1}) (x_k - x_{k+1}) + (R_1 - R_{n+1}) x_{n+1} \\ &= \sum_{k=1}^{n+1} R_k (x_k - x_{k-1}) - R_{n+1} x_{n+1} \end{aligned} \quad (2.2)$$

This implies that $\sum_{k=1}^{\infty} a_k x_k$ is convergent, then $a \in c_0^{\beta}(\Delta)$.

If $a \in c_0^{\beta}(\Delta)$ then $\sum_{k=1}^{\infty} a_k x_k$ is convergent for all $x \in c_0(\Delta)$. Obviously $a \in I_1$. If $x \in c_0(\Delta)$, then there

exists $y = (y_k) \in c_0$ such that $x_k = \sum_{i=1}^k y_i$ (1.1)

Then

$$\sum_{k=1}^n R_k y_k = \sum_{k=1}^n \left(\sum_{i=1}^k y_i \right) a_k + R_{n+1} \sum_{k=1}^n y_k \quad \text{with Abel summation formula. Thus we have}$$

$$\sum_{k=1}^n a_k x_k = \sum_{k=1}^n (R_k - R_{n+1}) y_k = \sum_{k=1}^n \left(\sum_{i=k}^n a_i \right) y_k \quad (2.3)$$

If we take

$$a_{nk} = \begin{cases} \sum_{i=k}^n a_i, & 1 \leq k \leq n \\ 0, & k > n \end{cases}$$

then $A = (a_{nk}) \in (c_0, c)$ since $\lim_n \sum_{k=1}^{\infty} a_{nk} y_k = \lim_n \sum_{k=1}^n a_{nk} y_k$ exists for all $y \in c_0$ (2.3). This implies that

$\sup_n \sum_{k=1}^{\infty} |a_{nk}| = \sup_n \sum_{k=1}^n \sum_{i=k}^n |a_i| < \infty$ [4]. Thus we get $\sum_{k=1}^{\infty} |R_k| < \infty$. Furthermore (2.2) implies that $\lim_n R_{n+1} x_{n+1}$ exists for each $x \in c_0(\Delta)$ then we get $(R_n) \in c_0^N(\Delta)$ by lemma 1. This completes the proof.

THEOREM 1. $A = (a_{nk}) \in (c_0(\Delta), c)$ iff

$T_1 \cdot (R_{nk}) \in c_0^N(\Delta)$, for each $n \in \mathbb{N}$

$T_2 \cdot R = (R_{nk}) \in (c_0, c)$

Proof. If $a \in (c_0(\Delta), c)$ then the series $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ are convergent for each $n \in \mathbb{N}$ and for all $x \in c_0(\Delta)$, this implies that $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and $\lim_n \sum_{k=p}^{\infty} a_{nk} = a_p$ exists for each $p \in \mathbb{N}$ [3] (page 166). From lemma 3 we have $\sum_{k=1}^{\infty} |R_{nk}| < \infty$, $\lim_k R_{nk} x_k = 0$ for each $n \in \mathbb{N}$ and for all $x \in c_0(\Delta)$. This proves T_1 . If we write again (2.2) we get

$$\sum_{k=1}^m a_{nk} x_k = \sum_{k=1}^{m+1} R_{nk} (x_k - x_{k-1}) - R_{n, m+1} x_{m+1} \quad (2.4)$$

and

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} R_{nk} (x_k - x_{k-1}) \quad (2.5)$$

This shows that $R \in (c_0, c)$. If we use again lemma 3 and (2.5) we get the sufficiency of T_1 and T_2 .

Similarly we can prove that

- i) $A \in (c_0(\Delta), c_0)$ iff T_1 and $R \in (c_0, c_0)$
- ii) $A \in (c_0(\Delta), l_{\infty})$ iff T_1 and $R \in (l_{\infty}, l_{\infty})$
- iii) $A \in (c_0(\Delta), M_0)$ iff T_1 , $R \in (l_{\infty}, l_{\infty})$ and

$$B = (b_{nk}) = (a_{nk} - a_{n, k+1}) \in (c_0(\Delta), c_0)$$

- iv) $A \in (c_0(\Delta), c_0(\Delta))$ iff $(a_{nk}) \in c_0^{\beta}(\Delta)$, for each $n \in \mathbb{N}$ and $C = (c_{nk}) = (a_{nk} - a_{n-1, k}) \in (c_0(\Delta), c_0)$ ($a_{0k} = 0$)

Open questions

1) Matrix maps for M_0 .

2) M_0 has a Schauder basis with $\|\cdot\|_{\Delta}$. It is (E_k) . (we can write $x = \sum_{k=1}^{\infty} (x_k - x_{k-1}) E_k$, each $x \in M_0$)

Then $(M_0, \|\cdot\|_{\Delta})$ is separable.

Is M_0 separable or have a Schauder basis with $\|\cdot\|_{\infty}$?

3) It is obvious that $c_0 \subset c \subset M_0 \subset l_{\infty}$ and inclusions are strict. In this order, is there a separable space E which is $c \subset E \subset l_{\infty}$ with the norm $\|\cdot\|_{\infty}$? If not, is c an upper bound according to separability?

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