COMPLEMENTED SUBSPACES OF p-ADIC SECOND DUAL BANACH SPACES

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ABSTRACT. Let K be a non-archimedean non-trivially valued complete field. In this paper we study Banach spaces over K. Some of main results are as follows:

1. The Banach space $BC((1^m)_1)$ has an orthocomplemented subspace linearly homeomorphic to $c_0$.
2. The Banach space $BC((c_0)_1)$ has an orthocomplemented subspace linearly homeomorphic to $1^m$.

KEY WORDS AND PHRASES. non-archimedean valued fields, non-archimedean (p-adic) Banach spaces, polar spaces, spherically complete, complemented subspaces.

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1. INTRODUCTION.

Throughout this paper K is a non-archimedean non-trivially valued complete field with a valuation $| |$, and $E, F$ are Banach spaces over K with a non-archimedean norm denoted by $\parallel \parallel$. Let $L(E,F)$ be the space consisting of all continuous linear maps of $E$ to $F$. The dual space of $E$ is $E' = L(E,K)$. The dual operator $T' \in L(F', E')$ of $T \in L(E,F)$ is defined as usual. If there exists a linear isometry from $E$ onto $F$, then $E$ and $F$ are said to be isomorphic and we denote $E \cong F$. For a Banach space $E$, if there exists a (ortho)complemented subspace of $F$ which is isomorphic to $E$, then $E$ is said to be (ortho)complemented in $F$. Let $S$ be a topological space and let $BC(S)$ be the Banach space consisting of all bounded continuous functions $S \rightarrow K$ with a norm

$$\parallel f \parallel = \sup\{ |f(s)| : s \in S \} \ (f \in BC(S)).$$

Let $E''$ be the second dual Banach space of $E$ and let $J_E : E \rightarrow E''$ be the natural map.

DEFINITION. If $J_E$ is linearly homeomorphic from $E$ into $E''$, then $E$ is said to be polar (see [6]).

DEFINITION. A Banach space $E$ is said to be strongly polar if every continuous seminorm $p$ on $E$ satisfies the following equality (see [7]).

$$p = \sup\{ |f| : f \in E', \ |f| \leq p \}$$

These spaces were first introduced by Schikhof [5] for locally convex topological spaces over $K$ and were studied by some authors (e.g. [1], [2]).

DEFINITION. Let $D$ be a subspace of $E$. If every $x' \in D'$ has an extension $\bar{x}' \in E'$, then $D$ has the weak extension property in $E$. In addition, if $\bar{x}'$ can be chosen such that $1x' = \bar{x}'1$, then we say that $D$ has the extension property in $E$.

For any $r > 0$ we put $E_r = \{ xeE : \|x\| < r \}$. Let $\pi$ denote an arbitrary fixed element of $K$ with $0 < |\pi| < 1$. Other terms will be used as in Rooij [4]. In this paper we deal with complemented subspaces of $BC((E')_1)$ and $E''$. Throughout this paper, we consider a subset $(E')_r \ (r > 0)$ of $E'$, $(E')_r$ is assumed to have the weak $\ast$ topology. In section 2 we show that there exists a Banach space $E$ such that $BC((1^m)_1)$ is linearly homeomorphic to $c_0 \oplus E$. And in section 3, we show that there exists a Banach space $F$ such that $BC((c_0)_1)$ is linearly homeomorphic to $1^m \oplus F$.
2. COMPLEMENTED SUBSPACES OF BC(S).

For every $T \in L(E, BC(S))$, for every $s \in S$ and for every $x \in E$, let

$$\psi_T(s)(x) = (T(x))(s). \quad (2.1)$$

Then the map $\psi_T(s)$ is a linear functional on $E$. Since $\|\psi_T(s)\| \leq \|T\|$, $\psi_T(s) \in (E')_{\text{compact}}$. Hence $\psi_T$ is a weak * continuous map from $S$ to $(E')_{\text{compact}}$. Conversely, for every weak * continuous map $\varphi : S \to (E')_{\text{r}}$ ($r > 0$), let

$$(T_\varphi(x))(s) = (\varphi(s))(x) \quad (x \in E, s \in S). \quad (2.2)$$

Then $T_\varphi(x)$ is a map from $S$ to $K$. Since for each $x \in E$

$$\sup\{|(T_\varphi(x))(s)| : s \in S, \|x\| \leq 1\}, \quad (2.3)$$

$T_\varphi(x) \in BC(S)$. Hence $T_\varphi$ is a linear map from $E$ to $BC(S)$. By (2.3), $\|T_\varphi\| \leq r$. It follows that $T_\varphi \in L(E, BC(S))$.

For the natural map $J_E : E \to E''$ and for every $x \in E$, let $R_E(x)$ denote the restriction of $J_E(x)$ to $(E')^\perp$, that is,

$$R_E(x) = J_E(x)|E'_{\perp}. \quad (2.4)$$

Then $R_E$ is a linear map from $E$ into $BC((E')^\perp)$. Since for every $x \in E$

$$\|R_E(x)\| = \sup\{|(R_E(x))(x')| : x' \in (E')^\perp\} \leq \sup\{\|x'\| \|x\| : x' \in (E')^\perp\} \|x\|, \quad (2.5)$$

we have $\|R_E\| \leq 1$ and $R_E \in L(E, BC((E')^\perp))$.

The next theorem follows from Schikhof [7].

THEOREM 1. Let $E$ be a strongly polar Banach space and let $D$ be a closed subspace of $E$. Then for each $e > 0$, each $f \in D'$ can be extended to an $f \in E$ with $\|f\| - e \|f\|$. (2.6)

A norm $\|\|_p$ on $E$ is said to be polar if

$$\|\|_p = \sup\{|f| : f \in E', |f| \leq 1\}. \quad (2.6)$$

We recall that if $E$ is polar, then there exists a polar norm $\|\|_p$ on $E$ such that it is equivalent to the original norm $\|\|$ (see [1, p. 75]), and so there exists a real number $d (d > 1)$ such that for every $x \in E$, $\|x\| \leq d\|x\|$. (2.7)

THEOREM 2. Let $E$ be a polar Banach space. Then there exists a real number $c (c > 1)$ satisfying the following (1) and (2).

(1) For each finite-dimensional subspace $D$ of $E$ and for each $f \in D'$ there exists an extension $\tilde{f} \in E'$ such that $\|\tilde{f}\| \leq c\|f\|$. (2.8)

(2) For each finite-dimensional subspace $D$ of $E$ there exists a projection $P : E \to D$ with $\|P\| \leq c$. (2.9)

PROOF. (1) Since $f \in D'$, it is trivial that $f(D, \|\|_p)'$. Let $c > 0$ be an arbitrarily given real number and put $c = (l + e)c$. By Theorem 2.1 in Garcia [1], there exists an extension $\tilde{f} \in E, \|\tilde{f}\|'$ such that $\|\tilde{f}\|' \leq (l + e)\|f\|$. Then we have that $\tilde{f} \|f\|\|f\| \leq d\|f\|$. (2.10)

(2) Using again Theorem 2.1 in [1], there exists a projection $P : E \to D$ such that $\|P\| \leq d\|P\|$. (2.11)

THEOREM 3. If $E$ is a polar space, then $R_E$ is a linear homeomorphism. And if the norm on $E$ is polar, then $R_E$ is a linear isometry.
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PROOF. In section 1, it is proved that for all $x \in E$

$$1 R_E(x) \leq 1 x 1.$$  (2.7)

Note that for every $x' \in E'$, $x' \neq 0$, there exists an integer $m$ with $|x'|^{m+1} \leq 1 x' 1 \leq |x'|^m$, then

$$|x'|^{m-1} \leq 1 x' 1 \leq |x'|^m.$$  (2.8)

From (2.7) and (2.8) it follows that

$$1 J_E(x) \leq 1 R_E(x) \leq 1 x 1.$$  (2.9)

Since $E$ is polar, $J_E$ is a homeomorphism, so is $R_E$. Next, if the norm $1$ of $E$ is polar, then for all $x \in E$ we have

$$1 x 1 = \sup \{|x'(x)| : x' \in E', 1 x' 1 \leq 1 x 1\}.$$  (2.10)

Therefore $R_E$ is a isometry.

COROLLARY 4. (1) For any strongly polar space $E$, $R_E$ is a linear isometry.

(2) For any topological space $S$, $R_{BC}(S)$ is a linear isometry.

THEOREM 5. For every $T \in L(E, BC(S))$, there exists a $\bar{T} \in L(BC((E')_1), BC(S))$ such that $\bar{T} \circ R_E = T$. In particular, if $T11 = 1$, then $\bar{T}$ satisfies $1 \bar{T}11 = 1$.

PROOF. At first, we notice that $(E')_1$ is supposed to carry the weak * topology. To show theorem, we may assume that $1 \bar{T}11 = 1$. Then $\psi_T$ is a weak * continuous map from $S$ into $(E')_1$. Define

$$\bar{T} : BC((E')_1) \to BC(S),$$  (2.11)

by

$$\bar{T}(f) = f \circ \psi_T \quad (f \in BC((E')_1)).$$  (2.12)

For every $x \in E$ and for every $s \in S$, we have

$$(T(R_E(x))(s) = (R_E(x))(\psi_T(s))(x) = (T(x))(s).$$  (2.13)

Then $\bar{T} \circ R_E = T$. Further,

$$1 \bar{T}11 = \sup \{|f(\psi_T(s))| : s \in S\} / |f|.$$  (2.14)

Hence if $1 \bar{T}11 = 1$, then

$$1 = 1 \bar{T}11 \leq 1 R_E \leq 1 \bar{T}11.$$  (2.15)

The proof is complete.

LEMMA 6. Let $E$, $F$ and $X$ be Banach spaces. Let $A : E \to X$ be a linear homeomorphism onto $X$ and $H : E \to F$ be a linear homeomorphism into $F$. If there exists an $A \in L(F, X)$ such that $A \circ H = A$, then the closed subspace $H(E)$ of $F$ is complemented. In particular, if $A$ and $H$ are linear isometries and $1 A1 = 1$, then $E$ is orthocomplemented in $F$.

PROOF. Put $P = H \circ A^{-1} : F \to H(E) \subset F$. Then $P$ is a projection onto $H(E)$. If $A$ and $H$ are linear isometries and $1 A1 = 1$, then $1 P11 = 1$. Hence $P$ is an orthoprojection.

THEOREM 7. Let $E$ be of countable type. Then $R_E(E)$ is complemented in $BC((E')_1)$.

Especially, $c_0$ is orthocomplemented in $BC((1')_1)$.
PROOF. If \( E \) is finite-dimensional, then the assertion of this theorem is clear. Hence we may assume \( E \) is infinite-dimensional. Since \( E \) is of countable type, \( E \) is a polar space. Then by Theorem 3 the map \( R_E : E \to BC((E')_1) \) is a linear homeomorphism into \( BC((E')_1) \). Further, since \( E \) is infinite-dimensional, for an infinite compact ultrametrizable space \( S \), \( E \) is linearly homeomorphic to \( BC(S) \) (see [4, p.190]). Let \( H_0 : E \to BC(S) \) be a linear homeomorphism onto \( BC(S) \). By Theorem 5, there exists an \( \tilde{H}_0 \in L(BC((c_0)_1), BC(S)) \) such that \( \tilde{H}_0 \circ R_E = H_0 \). Hence by Lemma 6, \( R_E(E) \) is complemented in \( BC((E')_1) \). If \( E = c_0 \), then the above \( H_0 \) can be taken as a linear isometry from \( c_0 \) onto \( BC(S) \). Since \( c_0 \) is strongly polar, by Corollary 4, the map \( R_{c_0} \) is linearly isometric. Hence by Theorem 5, there exists an \( \tilde{H}_0 \in L(BC(((c_0)_1)'), BC(S)) \) with \( \tilde{H}_0 I = I \). Thus, by Lemma 6, \( R_{c_0}(c_0) \) is orthocomplemented in \( BC(((c_0)_1)'). \) Since \( (c_0)_1' \sim l^\infty \), \( BC(((c_0)_1)'_1) \sim BC((l^\infty)_1) \). Hence \( c_0 \) is orthocomplemented in \( BC((l^\infty)_1) \).

The following corollary follows immediately from Theorem 7.

COROLLARY 8. Let \( E \) be of countable type. Then there exists a Banach space \( X \) such that \( BC((l^\infty)_1) \) and \( E \oplus X \) are linearly homeomorphic.

Since \( c_0 \) is linearly isometric to some \( BC(S) \), the second part of Theorem 7 is a special case of the following corollary.

COROLLARY 9. For any topological space \( S \), let \( E = BC(S) \). Then \( E \) is orthocomplemented in \( BC((E')_1) \).

PROOF. Let \( I : E \to BC(S) \) be the identity. Then there exists an \( \tilde{I} \in L(BC((E')_1), BC(S)) \) such that \( \tilde{I} \circ R_E = I \) and \( I I = I \). By Corollary 4, \( R_E(E) \) is linearly isometric. Put \( P = R_E \circ I^{-1} \circ \tilde{I} \). Then \( P \) is an orthoprojection of \( BC((E')_1) \) onto \( R_E(E) \). Hence \( E \) is orthocomplemented in \( BC((E')_1) \).

COROLLARY 10. The Banach space \( BC((c_0)_1) \) contains an orthocomplemented subspace linearly homeomorphic to \( l^\infty \). In particular if \( K \) is spherically complete, then the Banach space \( BC((c_0)_1) \) contains an orthocomplemented subspace linearly isometric to \( l^\infty \).

PROOF. Suppose that \( K \) is not spherically complete. Applying the extended version of Corollary 9 to \( S = \mathbb{N} \) (\( \mathbb{N} \) denotes the set of all natural numbers) and observing that \( E = l^\infty \) and \( E' = c_0 \), we can obtain this corollary. Furthermore, if \( K \) is spherically complete, then so is \( l^\infty \); it follows easily that the second part holds.

3. COMPLEMENTED SUBSPACES IN SECOND DUAL SPACES.

Let \( T \in L(E,F') \). Then \( T \) determines a map

\[
\phi_T : F \to E' \tag{3.1}
\]
defined by \( \phi_T(y)(x) = (T(x))(y) \quad (x \in E, \quad y \in F) \). Clearly, \( \phi_T \) is linear and \( \phi_T I = ITI \).

Hence \( \phi_T \in L(F,E'}). Let \( D \) be a closed subspace and let \( D^\perp \) be the annihilator of \( D \) in \( F' \), i.e. \( D^\perp = \{ x' \in F' : x'(d) = 0, \quad d \in D \} \). A subset \( A \) of \( E \) is said to be compactoid if for every \( \epsilon > 0 \), there exists a finite subset \( X \) of \( E \) such that \( A \subseteq B_E + C \), where \( B_E = \{ x \in E : \| x \| < \epsilon \} \) and \( C \) is the absolutely convex hull of \( X \). Let \( T(E_1) \) be compactoid in \( F \), then \( T \) is said to be compact. A Banach space \( E \) is said to be (0) -space if every \( T \in L(E,c_0) \) is compact.

PROPOSITION 11. Let \( E, F \) be Banach spaces and let \( D \) be a closed subspace of \( F \). Then for every \( T \in L(E,D^\perp) \), there exists a \( \overline{T} \in L(E',D^\perp) \) such that \( \overline{T} \circ J_E = T \) and \( I I = ITI \).

PROOF. Let \( J_{E'} : E' \to E'' \) be the canonical map. Define an operator
by \((\overline{T}(x''))(y) = (J_E(\phi_T(y))(x'')) \ (y \in F, x'' \in E'')\). For every \(x'' \in E''\), \(\overline{T}(x'')\) is a linear functional on \(F\) and \(\overline{T}(x'') \in \mathcal{T}I_1\), so \(\overline{T}(x'') \in F'\). For every \(y \in D\) and for every \(x \in E\),
\[
(\phi_T(y))(x) = (T(x))(y) = 0.
\]
(3.3)
Hence \((\overline{T}(x''))(y) = 0\). This means that \(\overline{T}(x'') \in D^4\). It follows that \(\overline{T} \in \mathcal{T}L(E'', D^4)\) and \(\mathcal{T}I_1 \in \mathcal{T}I_1\). Further, for every \(x \in E\) and for every \(y \in F\),
\[
((\overline{T} \ast J_E)(x))(y) = (J_E(\phi_T(y))(J_E(x))
= (J_E(x))(\phi_T(y))
= (\phi_T(y))(x)
= (T(x))(y).
\]
(3.4)
Hence \(\overline{T} \ast J_E = \overline{T}\). Therefore we have
\[
\mathcal{T}I_1 \ast \overline{T} \ast J_E = \overline{T}.
\]
(3.5)
Thus we complete the proof.

The following corollary is immediate from Proposition 11.

**COROLLARY 12.** Let \(E\) and \(F\) be Banach spaces. For every \(T \in \mathcal{T}L(E, F')\), there exists a \(T' \in \mathcal{T}L(E'', F')\) such that \(\overline{T} \ast J_E = T\) and \(\|T\| = \|T'\|\).

**PROOF.** In Proposition 11, put \(D = \{0\}\). Then \(D^4 = F'\).

**PROPOSITION 13.** Let \(E\) be a Banach space and let \(D\) be a closed subspace of \(E\). If \(D\) is linearly homeomorphic (resp. isometric) to some dual space and is complemented (resp. orthocomplemented) in \(E\), then \(J_E(D)\) is complemented (resp. orthocomplemented) in \(E''\). In particular, if \(K\) is not spherically complete and \(D\) is of countable type and complemented in \(E\), then \(J_E(D)\) is complemented in \(E''\).

**PROOF.** Let \(D\) be a complemented closed subspace of \(E\), linearly homeomorphic to a dual Banach space \(F'\). By Lemma 4.23, (ii) and (iii), in Rooij [4], \(J_D\) is a homeomorphism and there exists a projection of \(D''\) onto \(J_D(D)\), so there is a \(Q \in \mathcal{T}L(D'', D)\) with \(Q \ast J_D = J_D\) (= the identity map of \(D\)). As \(D\) is complemented in \(E\), there is a projection \(P : E \to D\). Then \(J_E \ast Q \ast P'' \in \mathcal{T}L(E'', J_E(D))\). As
\[
(Q \ast P'' \ast J_E \ast Q \ast P'' = Q \ast (J_D \ast P)
= (Q \ast J_D) \ast P = I_D \ast P = P,
\]
(3.6)
for \(x \in D\) we have
\[
(J_E \ast Q \ast P''(J_E(x)) = J_E(P(x)) = J_E(x),
\]
(3.7)
so \(J_E \ast Q \ast P''\) is the identity on \(J_E(D)\). Thus \(J_E \ast Q \ast P''\) is a projection of \(E''\) onto \(J_E(D)\).

If \(D\) is orthocomplemented in \(E''\) and linearly isometric to \(F'\), we obtain \(I_D I_1\) and \(I_1 I_{1} I_1\), whence \(I_E \ast Q \ast P'' I_1 I_1\). In particular, if \(K\) is not spherically complete and \(D\) is of countable type, then \(D\) is linearly homeomorphic to \((1^n)'\) or \(K^n\), where \(n\) is some positive integer. Hence by the first assertion of this proposition, we can complete the proof.

**COROLLARY 14.** Suppose \(K\) is not spherically complete. Let \(E\) be an infinite-dimensional polar space which is not a \((0)\)-space and let \(F\) be an infinite-dimensional Banach space of countable type. Then there exists a Banach space \(X\) such that \(E''\) is linearly homeomorphic to \(F \oplus X\).
PROOF. By hypothesis, there exists an infinite-dimensional complemented subspace $D$ of $E$ which is of countable type (see [6, p.23]). It follows from Proposition 13 that there exists a subspace $X$ of $E''$ such that $E''=J_E(D)\oplus X$. Since $E$ is a polar space, $J_E$ is a linear homeomorphism. Therefore, $J_E(D)$ is of countable type. Hence $J_E(D)$ and $F$ are linearly homeomorphic, so $E''$ is linearly homeomorphic to $F \oplus X$.

**COROLLARY 15.** The subspace $J_E(E)$ of $E''$ has the extension property in $E''$.

**PROOF.** For every continuous linear $x': J_E(E) \to K$ the function $\overline{x}=J_E(x')J_E$ is a continuous linear function $E'' \to K$ extending $x'$ and with $\|\overline{x}\|\leq\|x'\|$, hence $\|x'\|=\|\overline{x}\|$.

The following comment was given by the referee: From the proof of Corollary 15 we obtain a sort of "simultaneous extension", a linear isometry $x'\mapsto \overline{x}'$ of $(J_E(E))'$ onto $E''$ that assigns to every continuous linear function $J_E(E) \to K$ an extension $E'' \to K$. Further, the following question was asked by him: Under what circumstances is there an orthoprojection of $E''$ onto (the closure of) $J_E(E)$?

**COROLLARY 16.** Let $D$ be a closed subspace of $E$. If $J_D$ has an extension $T$ from $E$ into $D''$. Then $D$ has the weak extension property in $E$. In particular, if $\|T\|=\|J_D\|$, then $D$ has the extension property in $E$.

**PROOF.** By Corollary 12, for every $f\in D'$, there exists an $\tilde{f}\in D''$ such that $\tilde{f}=J_D f$ and $\|\tilde{f}\|=\|f\|$. Put $g=\tilde{f}+T$. Then $g\in E'$ and $g|D= f$. Hence $D$ has the weak extension property in $E$. If $\|T\|=\|J_D\|$, then by Corollary 12, for every $x\in E$

$$|g(x)|=|(\tilde{f}+T)(x)| \leq \|\tilde{f}\|+\|T\|\|x\|$$

$$=\|\tilde{f}\|\|J_D\|\|x\|+\|T\|\|x\|$$

Hence it holds that $\|g\|\leq \|\tilde{f}\|+\|T\|\|x\|$.

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