ABSTRACT. Let $X$ be a metric space and let $CB(X)$ denote the closed bounded subsets of $X$ with the Hausdorff metric. Given a complete subspace $Y$ of $CB(X)$, two fixed point theorems, analogues of results in [1], are proved, and examples are given to suggest their applicability in practice.

KEY WORDS AND PHRASES. Fixed Point Theorems
1980 AMS SUBJECT CLASSIFICATION CODE. 47H10; 54H25

Let $X$ be a metric space with metric $d$ and let $Y$ be a complete subspace of the space $CB(X)$ of all closed and bounded subsets of $X$, with the Hausdorff metric $p$:

$$p(A, B) = \max \{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \}. \quad (1)$$

In Hicks [1], fixed point theorems for set-valued maps $T : X \to CB(X)$ were proved; and illustrated with examples. We show that similar results for maps $T : Y \to X$ can be obtained, using essentially the same techniques as in Hicks [1].

**THEOREM 1.** Let $T : Y \to X$ be continuous. Then there is an $A \in Y$ such that $T(A) \subseteq A$ iff there exists a sequence $\{A_n\}_{n=0}^\infty$ in $Y$ with $T(A_n) \subseteq A_{n+1}$ (or $T(A_{n+1}) \subseteq A_n$) and

$$\sum_{n=0}^\infty p(A_n, A_{n+1}) < \infty. \quad (2)$$

In this case, $A_n \to A$ as $n \to \infty$. (In fact, we may let $A_{n+1} = A_n \cup \{T(A_n)\}$, for each $n$, for the case $T(A_n) \subseteq A_{n+1}$.)

**PROOF.** If $T(A) \subseteq A$, then we are done. Conversely, if the given conditions are met, then $\{A_n\}_{n=0}^\infty$ is Cauchy, so let $A \in Y$ be its limit. Thus $T(A_n) \to T(A)$. If $y \in A$, then

$$d(y, T(A)) \leq d(y, T(A_n)) + d(T(A_n), T(A)), \quad (3)$$

so

$$d(A, T(A)) \leq d(A, T(A_n)) + d(T(A_n), T(A)). \quad (4)$$
Since \(d(T(A_n), T(A)) \rightarrow 0\) and we have \(d(A, T(A_n)) \leq \rho(A, A_{n+1}) \rightarrow 0\), it follows that \(T(A) \in A\).

**Examples**

(1) Let \(X = \mathbb{R}\), with the usual metric. Define \(T : CB(\mathbb{R}) \rightarrow \mathbb{R}\) by

\[
T(A) = \alpha \sup(A) + (1 - \alpha) \inf(A),
\]

where \(\alpha \in [0,1]\). Then \(T\) is continuous. If \(A \in CB(\mathbb{R})\), then

\[
T(A \cup \{T(A)\}) = T(A) \in A \cup \{T(A)\}.
\]

(2) Let \(X = \mathbb{R}\) as in 1, and let \(r : [0, \infty) \rightarrow (0, \infty)\) be such that \(r - 1_{\mathbb{R}}\), where \(1_{\mathbb{R}}\) is the identity on \(\mathbb{R}\). Define \(T : CB(\mathbb{R}) \rightarrow \mathbb{R}\) by

\[
T(A) = \alpha r(\sup(A)) + (1 - \alpha)r(\inf(A)),
\]

where \(\alpha \in (0,1)\). Assuming \(r\) is continuous, so is \(T\). Let \(A_0 \in CB(\mathbb{R})\), and for \(n \in \mathbb{N}\), let

\[
A_{n+1} = A_n \cup \left[\inf \{T(A_k)\}, \sup \{T(A_k)\}\right].
\]

Theorem 1 yields \(A \in CB(\mathbb{R})\) with \(T(A) \in A\) if

\[
\sum_{n=1}^{\infty} \max \left\{d\left[\inf \{T(A_n)\}, A_n\right], d\left[\sup \{T(A_k)\}, A_n\right]\right\} < \infty.
\]

**Definition.** Let \((X, d)\) be a metric space and let \(Y\) be a subspace of \((CB(X), \rho)\). Let \(T : Y \rightarrow X\). Then \(T\) is **nice** if for each \(A \in Y\) and each \(x \in A\) with \(d(x, T(A)) = d(A, T(A))\), there exists a set \(B \in Y\) with \(T(B) = x\).

**Examples**

(3) Let \(X = \mathbb{R}^2\), \(T : CB(\mathbb{R}^2) \rightarrow \mathbb{R}^2\) defined by

\[
T(A) = \left(\inf (proj_1(A)), \sup (proj_1(A))\right).
\]

Let \(a > b\) and \(A = [0, a] \times [0, b]\). Then \(T(A) = (0, a)\), and \((0, b)\) is the only point of \(A\) whose distance from \((0, a)\) equals \(d(A, T(A))\). Let \(B = [0, b]^2\). Then \(T(B) = (0, b)\).

(4) Let \(X = \mathbb{R}^2\), and for \(A \in CB(\mathbb{R}^2)\), let \(T(A)\) be the center of the circle which circumscribes \(A\). Let \(r = d(A, T(A))\), and let \(x \in A\) with \(d(x, T(A)) = r\). Let \(B = A \cap B\left(x, \frac{\text{diam}(A)}{2}\right)\). Then \(T(B) = x\).

**Theorem 2.** Let \((X, d)\) be a metric space and let \(Y\) be a complete subspace of \((CB(X), \rho)\), each member of which is compact. Let \(T : Y \rightarrow X\) be continuous. Assume that \(K : [0, \infty) \rightarrow [0, \infty)\) is non-decreasing, \(K(0) = 0\), and

\[
\rho(A, B) \leq K\left(d(T(A), T(B))\right)
\]

for \(A, B \in Y\). If \(T\) is nice, then there is \(A \in Y\) such that \(T(A) \in A\) iff there exists \(A_0 \in Y\) for which
In this case, we can choose \( \{A_n\}_{n=1}^{\infty} \) such that \( T(A_{n+1}) \in A_n \) and \( A_n \to A \).

**Proof.** If \( T(A) \in A \), then we are done. If \( A_0 \in Y \) satisfies (\( \ast \)), let \( x_1 \in A_0 \) with \( d(x_1, T(A_0)) = d(A_0, T(A_0)) \). Since \( T \) is nice, let \( A_1 \in Y \) with \( T(A_1) = x_1 \).

Next, let \( x_2 \in A_1 \) with \( d(x_2, T(A_1)) = d(A_1, T(A_1)) \), and then let \( A_2 \in Y \) with \( T(A_2) = x_2 \). Then

\[
d(T(A_1), T(A_2)) = d(T(A_1), x_2)
\]
\[
= d(T(A_1), A_1) = d(x_1, A_1)
\]
\[
\leq \rho(A_0, A_1) \leq K(d(T(A_0), T(A_1)))
\]

so that

\[
K(d(T(A_1), T(A_2))) \leq K^2(d(T(A_0), T(A_1)))
\]
\[
= K^2(d(T(A_0), x_1))
\]
\[
= K^2(d(T(A_0), A_0)).
\]

Now, suppose we have \( x_n \in A_{n-1} \) and \( A_n \in Y \) with \( d(x_n, T(A_{n-1})) = d(A_{n-1}, T(A_{n-1})) \) and \( T(A_n) = x_n \). Let \( x_{n+1} \in A_n \) with \( d(x_{n+1}, T(A_n)) = d(A_n, T(A_n)) \) and let \( A_{n+1} \in Y \) with \( T(A_{n+1}) = x_{n+1} \). Then

\[
d(T(A_n), T(A_{n+1})) = d(T(A_n), x_{n+2})
\]
\[
= d(T(A_n), A_n) = d(x_n, A_n)
\]
\[
\leq \rho(A_{n-1}, A_n) \leq K(d(T(A_{n+1}), T(A_n))).
\]

so that

\[
K(d(T(A_n), T(A_{n+1}))) \leq K^2(d(T(A_{n-1}), T(A_n)))
\]
\[
= K(K(d(T(A_{n-1}), T(A_n))))
\]
\[
\leq K^2(d(T(A_{n-2}), T(A_{n-1})))
\]
\[
= K^3(d(T(A_{n-2}), T(A_{n-1})))
\]
\[
\leq \cdots \leq K^n(d(T(A_0), A_0)).
\]

Thus, since

\[
\rho(A_n, A_{n+1}) \leq K(d(T(A_n), T(A_{n+1}))).
\]

it follows from (\( \ast \)) that

\[
\sum_{n=0}^{\infty} \rho(A_n, A_{n+1}) < \infty,
\]

and then by Theorem 1, \( A_n \to A \) and \( T(A) \in A \). \( \blacksquare \)
Note that the conditions of theorem 2 force $T$ to be a bijection. In both of these theorems, we have used completeness of the given subspace $Y$ of $CB(X)$ instead of completeness of $X$. In fact, in theorem 2, since $T$ is a bijection, we may trade completeness of $Y$ back for completeness of $X$ and use the second theorem of Hicks [1].

**THEOREM 3.** If $(X,d)$ is a complete metric space and $Y$ is any subspace of $(CB(X),p)$, each member of which is compact, then for any homeomorphism $T: Y \to X$ such that

$$p(A, B) \leq K(d(T(A), T(B))), \tag{18}$$

where $K : [0, \infty) \to [0, \infty)$ is nondecreasing, with $K(0) = 0$, there is $A \in Y$ such that $T(A) \in A$ iff there exists $A_0 \in Y$ for which (*) holds.

**PROOF.** If $A_0 \in Y$ satisfies (*), let $x_0 = T(A_0)$. Apply theorem 2 of Hicks [1] to $T^{-1} : X \to Y$ to obtain a $p \in X$ such that $p \in T^{-1}(p)$. Let $A = T^{-1}(p)$. Then $T(A) = p$ is in $A$, so we are done. ■

**REFERENCES**

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