EXISTENCE OF PERIODIC SOLUTIONS FOR NONLINEAR LIENARD SYSTEMS

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ABSTRACT. We prove the existence and multiplicity of periodic solutions for nonlinear Lienard System of the type

\[ x''(t) + \frac{d}{dt} [\nabla F(x(t))] + g(x(t)) + h(t,x(t)) + e(t) \]

under various conditions upon the functions g, h and e.

KEY WORDS AND PHRASES: Nonlinear Lienard system, multiplicity of periodic solution.

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1. INTRODUCTION

Let \( \mathbb{R}^n \) be \( n \)-dimensional Euclidean space. We define \( \| x \| = (\sum_{i=1}^{n} |x_i|^2)^{1/2} \) for \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \). By \( L^2([0, 2\pi], \mathbb{R}^n) \) we denote the space of all measurable functions \( x: [0, 2\pi] \rightarrow \mathbb{R}^n \) for which \( \| x(t) \|^2 \) is integrable. The norm is given by

\[ \| x \|_{L^2} = \left( \sum_{i=1}^{n} \| x_i \|_{L^2}^2 \right)^{1/2}. \]

By \( C^k([0, 2\pi], \mathbb{R}^n) \) we denote the Banach space of 2\( \pi \)-periodic continuous functions \( x: [0, 2\pi] \rightarrow \mathbb{R}^n \) whose derivatives up to order \( k \) are continuous. The norm is given by

\[ \| x \|_{C^k} = \sum_{i=0}^{k} \| x^{(i)} \|_{L^\infty}. \]

where \( \| y \|_{C^k} = \sup_{t \in [0, 2\pi]} \| y(t) \| \) which is a norm in \( C([0, 2\pi], \mathbb{R}^n) \). We use the symbol \( (\cdot, \cdot) \) for the Euclidean inner product in the space \( \mathbb{R}^n \). For \( x, y \) in \( C([0, 2\pi], \mathbb{R}^n) \) we define the \( L^2 \)-inner product as follows

\[ (x, y) = \int_0^{2\pi} (x(t), y(t)) dt. \]

The mean value \( \overline{x} \) of \( x \) and the function of mean value zero are defined by \( \overline{x} = \frac{1}{2\pi} \int_0^{2\pi} x(t) dt \) and \( \overline{x}(t) = x(t) - \overline{x} \), respectively.

We define inequalities in \( \mathbb{R}^n \) componentwise, i.e. \( x, y \in \mathbb{R}^n, x \leq y \) if and only if \( x_i \leq y_i \) for \( i = 1, 2, \ldots, n \), and \( x < y \) if and only if \( x_i < y_i \) for \( i = 1, 2, \ldots, n \). In this work, we will study the existence of periodic solutions and multiple periodic solutions for the problem

\[ (E) \quad x''(t) + \frac{d}{dt} [\nabla F(x(t))] + g(x(t)) + h(t,x(t)) + e(t) = 0 \]

\[ (B) \quad x(0) = x(2\pi) = x'(0) = x'(2\pi) = 0 \]
where $F : \mathbb{R}^n \to \mathbb{R}$ is a $C^2$-function, $g : \mathbb{R}^n \to \mathbb{R}$ is continuous, $h : [0, 2\pi] \times \mathbb{R}^n \to \mathbb{R}$ is continuous in both variables and $2\pi$-periodic in $t$, and $e : [0, 2\pi] \to \mathbb{R}$ is in $L^1([0, 2\pi], \mathbb{R})$. We assume that $g(x) = (g_1(x), g_2(x), \ldots, g_n(x))$ for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $h(t, x) = (h_1(t, x), h_2(t, x), \ldots, h_n(t, x))$ for all $(t, x) \in [0, 2\pi] \times \mathbb{R}^n$.

Moreover, we assume the following:

$(H_1)$ $h$ is bounded; i.e., for each $i = 1, 2, 3, \ldots, n$, there exists $K_i > 0$ such that
$$|h_i(t, x)| \leq K_i$$
for all $(t, x) \in [0, 2\pi] \times \mathbb{R}^n$.

$(H_2)$ for each $i = 1, 2, \ldots, n$,
$$\frac{d}{dt} \frac{\partial F(x)}{\partial x_i} = \frac{\partial^2 F(x)}{\partial x_i^2} x_i'$$
and there exists $C_i > 0$ such that
$$\left| \frac{\partial^2 F(x)}{\partial x_i^2} \right| \leq C_i$$
for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

The purpose of this work is to give existence and multiplicity results for periodic solutions of coupled Liénard system in $\mathbb{R}^n$. This paper was motivated by the results in [1] and so our results in this work extend some results in [1]. To prove our results we adapt Mawhin's continuation theorem in [2], and we give appropriate region for the system's multiplicity by finding an a'priori bound.

2. A'priori Bound

To prove our assertion, we consider the following homotopy:

$$(E_\lambda) \quad x''(t) + \lambda \frac{d}{dt} [VF(x(t))] + \lambda g(x) + \lambda h(t, x) + \lambda e(t) = 0.$$ 

Let $\lambda \in (0, 1)$ and let $x(t)$ be a possible solution of the problem $(E_\lambda)(B)$. Taking $L^2$-inner product by $x'(t)$ on both sides of $(E_\lambda)$, we have

$$\lambda \sum_{i=1}^n \int_0^{2\pi} \frac{\partial^2 F(x(t))}{\partial x_i^2} [x_i'(t)]^2 dt + \lambda \sum_{i=1}^n \int_0^{2\pi} g_i(x(t)) x_i'(t) dt$$
$$+ \lambda \sum_{i=1}^n \int_0^{2\pi} h_i(t, x(t)) x_i'(t) dt = \lambda \sum_{i=1}^n \int_0^{2\pi} e_i(t) x_i'(t) dt .$$

By the continuity of $\frac{\partial^2 F(x)}{\partial x_i^2}$, $(H_2)$ and the periodicity of $x_i(t)$ in $t$, we have

$$\sum_{i=1}^n C_i \int_0^{2\pi} [x_i'(t)]^2 dt \leq \left( \sum_{i=1}^n \int_0^{2\pi} \frac{\partial^2 F(x)}{\partial x_i^2} [x_i'(t)]^2 dt \right)^{1/2}$$
$$\quad \leq \sum_{i=1}^n \sqrt{2\pi} \left( \sum_{i=1}^n K_i \right)^{1/2} \left[ \int_0^{2\pi} [x_i'(t)]^2 dt \right]^{1/2} + \sum_{i=1}^n \left[ \int_0^{2\pi} [x_i'(t)]^2 dt \right]^{1/2} \left[ \sum_{i=1}^n \left[ \int_0^{2\pi} [x_i'(t)]^2 dt \right] \right]^{1/2} .$$

Hence

$$\|x\|_{L^2} \leq \left( \frac{1}{\min_{1 \leq i \leq n} C_i} \right) \left[ \sqrt{2\pi} \sum_{i=1}^n K_i \right]^{1/2} + \|e\|_{L^2} = M_0 .$$

By the Sobolev inequality, we have

$$\|x\|_{L^\infty} \leq \sqrt{\frac{\pi}{6}} M_0 = M_1 .$$
Suppose there exist \(a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n)\) in \(\mathbb{R}\) such that \(a \leq b\); if \(x(t)\) is a solution of \((E)(B)\) such that \(a \leq x(t) \leq b\) and \(\|x\| = M_i\), then
\[
\|x\| = \left[ \sum_{i=1}^{n} \left( \max(|a_i|, |b_i|)^2 \right) \right]^{1/2} + M_i.
\]

Taking \(L^2\)-inner product by \(x''(t)\) on both sides of \((E)\), we have
\[
\frac{1}{2} \int_0^{2\pi} [x''(t)]^2 \, dt + \lambda \sum_{i=1}^{n} \int_0^{2\pi} \frac{\partial^2 F(x)}{\partial x_i^2} x_i'(t)x_i''(t) \, dt
\]
\[
+ \lambda \sum_{i=1}^{n} \int_0^{2\pi} g_i(x_i(t))x_i''(t) \, dt + \lambda \sum_{i=1}^{n} \int_0^{2\pi} h_i(t,x(t))x_i''(t) \, dt
\]
\[
= \lambda \sum_{i=1}^{n} \int_0^{2\pi} \epsilon_i(t)x_i''(t) \, dt.
\]

Since \(F\) is a \(C^2\)-function, for each \(i = 1, 2, \ldots, n\), there exists \(\epsilon > 0\) such that
\[
\frac{\partial^2 F(x)}{\partial x_i^2} \leq D_i,
\]
and also since \(g\) is continuous, for each \(i = 1, 2, \ldots, n\), there exists \(L_i > 0\) such that
\[
|g_i(x_i)| \leq L_i.
\]

Hence
\[
\sum_{i=1}^{n} \int_0^{2\pi} [x''(t)]^2 \, dt \leq \left( \max_{i=1}^{n} D_i \right) \left[ \sum_{i=1}^{n} \int_0^{2\pi} |x_i'(t)|^2 \, dt \right]^{1/2} \left[ \sum_{i=1}^{n} \int_0^{2\pi} |x_i''(t)|^2 \, dt \right]^{1/2}
\]
\[
+ \sqrt{2\pi} \left[ \sum_{i=1}^{n} L_i^2 \right]^{1/2} + \left[ \sum_{i=1}^{n} K_i^2 \right]^{1/2} \left[ \sum_{i=1}^{n} \int_0^{2\pi} |x_i''(t)|^2 \, dt \right]^{1/2}
\]
\[
+ \left[ \sum_{i=1}^{n} \int_0^{2\pi} |\epsilon_i(t)|^2 \, dt \right]^{1/2} \left[ \sum_{i=1}^{n} \int_0^{2\pi} x_i''(t)^2 \, dt \right]^{1/2}
\]
and thus we have
\[
\|x''\|_{L^2} \leq \left( \max_{i=1}^{n} D_i \right) M_0 + \sqrt{2\pi} \left[ \sum_{i=1}^{n} L_i^2 \right]^{1/2} + \left[ \sum_{i=1}^{n} K_i^2 \right]^{1/2} + \|\epsilon\|_{L^2} = M_2.
\]

By the Sobolev inequality
\[
\|x'\| = \sqrt{\frac{\pi}{6}} M_2
\]
for every solution of the problem \((E)(B)\) where \(M_2\) depends on \(a, b, M_0\) and \(M_i\).

3. **OPERATOR FORMULATION**

Define
\[
L: D(L) \subseteq C^1([0, 2\pi], \mathbb{R}^n) \to L^2([0, 2\pi], \mathbb{R}^n)
\]
by
\[
(x_1(t), x_2(t), \ldots, x_n(t)) \to (x_1''(t), x_2''(t), \ldots, x_n''(t))
\]
where \(D(L) = C^2([0, 2\pi], \mathbb{R}^n)\). Then \(\text{Ker}L = \mathbb{R}^2\) and
Consider two continuous projections

\[ P : C^1([0, 2\pi], R^n) \to C^1([0, 2\pi], R^n) \]

such that

\[ \text{Im} P = \text{Ker} L \]

and

\[ Q : L^2([0, 2\pi], R^n) \to L^2([0, 2\pi], R^n) \]

defined by

\[ (Qe)(t) = \frac{1}{2\pi} \int_0^{2\pi} e(t)dt \]

Then

\[ \text{Ker} Q = \text{Im} L, C([0, 2\pi], R^n) = \text{Ker} L \oplus \text{Im} Q \]

and \( L^2([0, 2\pi], R^n) = \text{Im} L \oplus \text{Im} Q \) as a topological sum. Since

\[ \dim [L^2([0, 2\pi], R^n) / \text{Im} L] = \dim [\text{Im} Q] = \dim [\text{Ker} L] = n \]

\( L \) is a Fredholm mapping of index zero and hence there exists an isomorphism \( J : \text{Im} Q \to \text{Ker} L \). The operator \( L \) is not bijective but the restriction of \( L \) on \( \text{Dom} L \cap \text{Ker} P \) is one-to-one and onto \( \text{Im} L \), so it has its algebraic right inverse \( K_\text{ap} \) and, as well known, it is compact. Define

\[ N : C^1([0, 2\pi], R^n) \to L^2([0, 2\pi], R^n) \]

by

\[ x(t) \to -\frac{d}{dt} [\nabla F(x(t))] - g(x(t)) - h(t, x(t)) + e(t) \]

where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \). Then \( N \) is continuous and maps bounded sets into bounded sets. Let \( G \) be any open bounded subset of \( C^1([0, 2\pi], R^n) \), then \( QN : G \to L^2([0, 2\pi], R^n) \) is bounded and \( K_\text{ap}(I - Q) : G \to L^2([0, 2\pi], R^n) \) is compact and continuous. Hence \( N \) is \( L \)-compact on \( G \). Now we see \( x \in D(L) \) is a solution to the problem \((E_\lambda)(B)\) if and only if

\[ Lx = \lambda Nx \]

4. MAIN RESULTS

**THEOREM 4.1.** Besides conditions on \( F, g, e, \) and \((H_1), (H_2)\), we assume

\((H_3)\) there exists \( r = (r_1, r_2, \ldots, r_n), s = (s_1, s_2, \ldots, s_n), A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) in \( R^n \) such that \( r < s \) and \( A \leq B \)

\[ \frac{1}{2\pi} \int_0^{2\pi} g(r + \bar{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t))dt \leq A \]

and

\[ \frac{1}{2\pi} \int_0^{2\pi} g(s + \bar{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t))dt \geq B \]

for every \( \bar{x} \in R^n \) such that

\[ \| \bar{x} \| \leq \left[ \sum_{i=1}^{n} [\max(|r_i|, |s_i|)^2]^{1/2} \right]^{1/2} \]
and for every $\bar{x} \in C^1([0, 2\pi], \mathbb{R}^n)$ having mean value zero, satisfying the boundary condition (B) and such that

$$\|\bar{x}\|_{L^2} \leq \sqrt{\frac{\pi}{6} \left( \frac{1}{\min_{i \neq \infty} C_i} \right) \left[ \sqrt{2\pi \left( \sum_{i=1}^{n} K_i^2 \right)} + \|\bar{e}\|_{L^2} \right]} = M_0$$

Then $(E)(B)$ has at least one solution if

$$A < \frac{1}{2\pi} \int_0^{2\pi} e(t) dt < B.$$ 

**PROOF.** We construct a bounded open set $\Omega$ in $C^1([0, 2\pi], \mathbb{R}^n)$ to apply Mawhin's continuation theorem in [2]. Using a priori estimate, we have

$$\|x\|_{L^1} \leq \left( \frac{1}{\min_{i \neq \infty} C_i} \right) \left[ \sqrt{2\pi \left( \sum_{i=1}^{n} K_i^2 \right)} + \|\bar{e}\|_{L^2} \right] = M_0$$

for any solution $x(t)$ of $(E_0)(B), \lambda \in (0, 1)$. Hence $\|\bar{x}\|_{L^2} \leq \sqrt{\frac{2}{6} M_0} = M_1$. Define a bounded set $\Omega'$ by

$$\Omega' = \{ x \in C^1([0, 2\pi], \mathbb{R}^n) \mid r < \bar{x} < s, \|x\|_{L^\infty} < M_1 \}.$$ 

Then, for any solution $x(t)$ of $(E_0)(B)$ lying in $\Omega'$, we have

$$\|x\|_{L^1} \leq \left[ \sum_{i=1}^{n} \left( \max_{i \neq \infty} (r_i, s_i) \right) \right]^{1/2} + M_1$$

and

$$\|x''\|_{L^2} \leq \left( \max_{i \neq \infty} D_i \right) M_0 + \sqrt{2\pi} \left[ \sum_{i=1}^{n} L_i^2 \right]^{1/2} + \left[ \sum_{i=1, i \neq \infty} K_i^2 \right]^{1/2} + \|\bar{e}\|_{L^2} = M_2,$$

where $L_i$ depends on $r, s$ and $M_1$. Thus $\|x\|_{L^\infty} \leq \sqrt{\frac{2}{6} M_2}$. Define a bounded open set $\Omega$ by

$$\Omega = \left\{ x \in C^1([0, 2\pi], \mathbb{R}^n) \mid r < \bar{x} < s, \|x\|_{L^\infty} < 2M_1, \|x''\|_{L^1} < \sqrt{\frac{2}{6} M_2} \right\}.$$ 

Let $(x, \lambda) \in [D(L) \cap \partial \Omega] \times (0, 1)$ and if $(x, \lambda)$ is any solution to $Lx = \lambda Nx$, then $(x, \lambda)$ is a solution to the problem $(E_0)(B)$,

$$\|\bar{x}\| = \left[ \sum_{i=1, i \neq \infty} \left( \max_{i \neq \infty} (r_i, s_i) \right) \right] \leq M_1$$

and there exists some $i \in \{1, 2, \ldots, n\}$ such that $\bar{x}_i = r_i$ or $s_i$. Take $L^2$-inner product with $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ on both sides of $(E_2)$, we have

$$\lambda \int_0^{2\pi} g_i(x_i(t)) dt + \lambda \int_0^{2\pi} h_i(t, x(t)) dt = \lambda \int_0^{2\pi} e_i(t) dt,$$

or

$$\int_0^{2\pi} g_i(x_i(t)) dt + \int_0^{2\pi} h_i(t, x(t)) dt = \int_0^{2\pi} e_i(t) dt = 0$$

if $\bar{x}_i = r_i$, then, by assumption

$$\int_0^{2\pi} g_i(r_i + \bar{x}_i(t)) dt + \int_0^{2\pi} h_i(t, \bar{x}_i(t), \ldots, r_i + \bar{x}_i(t), \ldots, \bar{x}_n(t)) dt - \int_0^{2\pi} e_i(t) dt < 0.$$ 

If $\bar{x}_i = s_i$, then again by assumption,
Thus, for each \( \lambda \in (0,1) \), for every solution of
\[
Lx = \lambda Nx
\]
is such that \( x \notin \partial \Omega \).

Next, we will show that \( QNx \neq 0 \) for each \( x \in \text{Ker}L \cap \partial \Omega \) and \( d_{\partial}[JQN, \Omega \cap \text{Ker}L, 0] \neq 0 \)
where \( d_{\partial} \) is the Brouwer topological degree. Since \( J: \text{Im}Q \rightarrow \text{Ker}L \) is an isomorphism and
\[
\dim[\text{Im}Q] = \dim[\text{Ker}L] = n,
\]
we may take \( J \) to be the identity on \( \mathbb{R}^n \) and hence
\[
(JQN)(x)(t) = -\frac{1}{2\pi} \int_0^{2\pi} g(x(t))dt - \frac{1}{2\pi} \int_0^{2\pi} h(t,x(t))dt + \frac{1}{2\pi} \int_0^{2\pi} e(t)dt
\]
with, for \( i = 1, 2, \ldots, n \),
\[
(JQN)_i(x)(t) = -\frac{1}{2\pi} \int_0^{2\pi} g_i(x(t))dt - \frac{1}{2\pi} \int_0^{2\pi} h_i(t,x(t))dt + \frac{1}{2\pi} \int_0^{2\pi} e_i(t)dt
\]
where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \).

Let \( x \in \text{Ker}L \cap \partial \Omega \), then \( x = \overline{x} \) is constant in \( \mathbb{R}^n \),
\[
\| \overline{x} \| \leq \left[ \sum_{i=1}^{n} \max(|r_i|, |s_i|)^2 \right]^{1/2},
\]
and there exists \( i \in \{1, 2, \ldots, n\} \) such that \( x_i = \overline{x}_i = r_i \) or \( s_i \). In a similar manner we have \( (QN)_i(x) \neq 0 \).

Thus \( QNx \neq 0 \) for each \( x \in \text{Ker}L \cap \partial \Omega \). It is easy to see that \( P = \overline{\Omega} \cap \text{Ker}L = \bigcap_{i=1}^{n} [r_i, s_i] \). Let
\[
P_i = \{ x \in P \mid x_i = r_i \}, \quad P_i' = \{ x \in P \mid x_i = s_i \}, \quad x \in P, \quad x' \in P', \quad i = 1, 2, \ldots, n.
\]

Then \( x = \overline{x}, x' = \overline{x}' \) are constant with
\[
\| \overline{x} \|, \quad \| \overline{x}' \| \leq \left[ \sum_{i=1}^{n} \max(|r_i|, |s_i|)^2 \right]^{1/2},
\]
and \( x_i = \overline{x}_i = r_i, x_i' = \overline{x}_i' = s_i \). Hence
\[
(JQN)_i(x) = -\frac{1}{2\pi} \int_0^{2\pi} g_i(r_i)dt - \frac{1}{2\pi} \int_0^{2\pi} h_i(t, \overline{x}, \ldots, r_i, \ldots, x_n)dt + \frac{1}{2\pi} \int_0^{2\pi} e_i(t)dt > 0
\]
and
\[
(JQN)_i(x') = -\frac{1}{2\pi} \int_0^{2\pi} g_i(s_i)dt - \frac{1}{2\pi} \int_0^{2\pi} h_i(t, x_1', \ldots, s_i, \ldots, x_n')dt + \frac{1}{2\pi} \int_0^{2\pi} e_i(t)dt < 0.
\]
Thus \( (JQN)_i(x)(JQN)_i(x') < 0 \) for \( i = 1, 2, \ldots, n \). Therefore, by the generalized intermediate value theorem, \( d_{\partial}[JQN, \Omega \cap \text{Ker}L, 0] \neq 0 \). Hence, by Mawhin's continuation theorem, the problem \( (E)(B) \) has at least one solution in \( D(L) \cap \overline{\Omega} \).

**THEOREM 4.2.** Besides conditions on \( F, g, e, \) and \( (H_1) \) and \( (H_2) \), we assume

\((H_3)\) there exists \( q = (q_1, q_2, \ldots, q_n), \quad r = (r_1, r_2, \ldots, r_n), \quad s = (s_1, s_2, \ldots, s_n), \quad A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) in \( \mathbb{R}^n \) such that \( q < r < s \) and \( A \leq B \) such that
\[
\frac{1}{2\pi} \int_0^{2\pi} g(q + \overline{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \overline{x} + \overline{x}(t))dt \geq B,
\]
\[
\frac{1}{2\pi} \int_0^{2\pi} g(r + \overline{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \overline{x} + \overline{x}(t))dt \leq A,
\]
and
\[ \frac{1}{2\pi} \int_0^{2\pi} g(x + \ddot{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, x + \ddot{x}(t))dt = B \]
for every $\ddot{x} \in \mathbb{R}^n$ such that
\[ \|\ddot{x}\| = \left( \sum_{i=1}^{n} \max(|q_i|, |r_i|, |s_i|)^2 \right)^{1/2} \]
and for every $x \in C^1([0,2\pi], \mathbb{R}^n)$ having mean value zero, satisfying the boundary condition $(B)$ such that
\[ \|x\| = \sqrt{\frac{n}{6}} \left( \min_{1 \leq i \leq n} C_i \right) \left( \sqrt{2\pi} \left( \sum_{i=1}^{n} K_i \right)^{1/2} + \|\ddot{x}\|_2 \right) \]
Then $(E)(B)$ has at least $2^n$ solutions if
\[ A < \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e(t)dt < B. \]

**PROOF.** We construct $2^n$ bounded open sets in $C^1([0,2\pi], \mathbb{R}^n)$ to apply Mawhin's continuation theorem in [3]. Using a priori estimate, we have
\[ \|x\|_L^2 = \left( \frac{1}{\min_{1 \leq i \leq n} C_i} \right) \left[ \sqrt{2\pi} \left( \sum_{i=1}^{n} K_i \right)^{1/2} + \|\ddot{x}\|_2 \right] = M_0 \]
for any solution $x(t)$ of $(E_\lambda)(B), \lambda \in (0,1)$. Hence $\|\ddot{x}\|_2 = \sqrt{\frac{\pi}{6}} M_0 = M_1$. Let $I, J$ be two disjoint subsets of \{1, 2, ..., n\} such that $I \cup J = \{1, 2, ..., n\}$ and define $\Omega_{ij}^\lambda$ by $\Omega_{ij}^\lambda = \{ x \in C^1([0,2\pi], \mathbb{R}^n) \mid q_i \leq x_i \leq r_i \}
for i \in I, r_j \leq x_j \leq s_j \}
for j \in J, \|x\|_2 = M_1 \};$ then the number of such sets is $2^n$ and for any solution $x(t)$ of $(E_\lambda)(B)$ lying in $\Omega_{ij}^\lambda$, we have
\[ \|x\|_2 \leq \left( \sum_{i \in I} \left[ \max(|q_i|, |r_i|) \right]^2 + \sum_{j \in J} \left[ \max(|r_j|, |s_j|) \right]^2 \right)^{1/2} + M_1 \]
and
\[ \|x''\|_L^2 = \left( \max_i D_i \right) M_0 + \sqrt{2\pi} \left( \sum_{i=1}^{n} L_i^2 \right)^{1/2} + \left( \sum_{i=1}^{\infty} K_i \right)^{1/2} + \|\ddot{x}\|_2 = M_2 \]
where $L_i$ depends on $q_i, r_i, s_i$ and $M_1$. Thus $\|x\|_2 \leq \sqrt{\frac{2\pi}{3}} M_2$. Define a bounded open set $\Omega_{ij}$ by
\[ \Omega_{ij} = \{ x \in C^1([0,2\pi], \mathbb{R}^n) \mid q_i \leq x_i \leq r_i \}
for i \in I, r_j \leq x_j \leq s_j \}
for j \in J, \|x\|_2 < 2M_2, \|x''\|_2 < \sqrt{\frac{2\pi}{3}} M_2 \}.
Let $(x, \lambda) \in [D(L) \cap \partial \Omega_{ij}] \times (0,1)$ and if $(x, \lambda)$ is any solution to
\[ Lx = \lambda Nx, \]
then $(x, \lambda)$ is a solution to the problem $(E_\lambda)(B)$,
\[ \|x\| \leq \left( \sum_{i \in I} \left[ \max(|q_i|, |r_i|) \right]^2 + \sum_{j \in J} \left[ \max(|r_j|, |s_j|) \right]^2 \right)^{1/2}, \|\ddot{x}\| \leq M_1 \]
and there exists some $i \in \{1, 2, ..., n\}$, such that $x_i = q_i, r_i$ or $s_i$. By $(H_2)$ and assumption we can see for each $\lambda \in (0,1)$, for every solution of $Lx = \lambda Nx$ is such that $x \notin \partial \Omega_{ij}$. And similarly, we can also see $Qx \neq 0$ for each $x \in KerL \cap \partial \Omega_{ij}$. It is easy to see $P = \Omega \cap KerL = \Pi_{i \in I}[q_i, r_i] \times \Pi_{j \in J}[r_j, s_j]$. Let
\[ P_i = \{ x \in p \mid x_i = q_i \} \quad \text{if} \quad i \in I, \]
\[ P_j = \{ x \in p \mid x_j = r_j \} \quad \text{if} \quad j \in J, \]
\[ P_i' = \{ x \in p \mid x_i = r_i \} \quad \text{if} \quad i \in I, \]
\[ P_j' = \{ x \in p \mid x_j = s_j \} \quad \text{if} \quad j \in J, \]
and let \( x \in P_i, x' \in P_i' \) with \( i \in I \cup J \). Then, for \( i \in I \), we have \( x_i = q_i, x_i = r_i \). Hence
\[ (JQN)_i(x)(JQN)_i(x') < 0 \quad \text{for} \quad i \in I. \]
For \( j \in J \), we have \( x_j - r_j, x_j' - s_j \). Thus \( (JQN)_j(x)(JQN)_j(x') < 0 \) for \( j \in J \). Therefore, we have \( d_\gamma[JQN, \Omega \cap KerL, 0] \neq 0 \). Thus, by Mawhin’s continuation theorem, the problem \( (E_\gamma)(B) \) has at least one solution in \( D(L) \cap \Omega_L \). Thus \( (E_\gamma)(B) \) has at least \( 2^* \) solutions.

**Corollary 4.3.** Besides the conditions on \( F, g, e, \) and \( (H_1) \) and \( (H_2) \), we assume

\( (H_3) \) there exists \( T = (T_1, T_2, \ldots, T_n) > 0 \) in \( R^* \) such that
\[ g(T + x) - g(x) \quad \text{and} \quad h(t, T + x) - h(t, x) \]
for all \( (t, x) \in [0, 2\pi] \times R^* \).

\( (H_4) \) there exists \( r = (r_1, r_2, \ldots, r_n), s = (s_1, s_2, \ldots, s_n), A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) in \( R^* \) such that \( 0 < s - r < T, r < s, A < B \)
\[ \frac{1}{2\pi} \int_0^{2\pi} g(s + \bar{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x}(t))dt \leq A, \]
\[ \frac{1}{2\pi} \int_0^{2\pi} g(s + \bar{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x}(t))dt \geq B \]
for every \( \bar{x} \in R^* \) such that
\[ \|\bar{x}\| = \sqrt{\sum_{i=1}^{n} \left[ \max(|s_i - T_i|, |r_i|, |s_i|) \right]^2} \]
and for every \( \bar{x} \in C^1([0, 2\pi], R^*) \) having mean value zero, satisfying the boundary condition \( (B) \) and such that
\[ \|\bar{x}\| = \sqrt{\frac{\pi}{6} \left( \frac{1}{\min_{x \in [a, b]} C_i} \right) \left[ \sqrt{2\pi} \left( \sum_{k=1}^{n} K_k^2 \right)^{1/2} + \|e\|_{L^2} \right].} \]
Then \( (E)(B) \) has at least \( 2^* \) solutions if
\[ A < \frac{1}{2\pi} \int_0^{2\pi} e(\bar{x})(t)dt < B. \]

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**REFERENCES**


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