A REMARK ON CERTAIN p-VALENT FUNCTIONS

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ABSTRACT. The object of the present paper is to prove an interesting result for certain analytic and p-valent functions in the unit disc U = \{z: |z| < 1\}.

KEY WORDS AND PHRASES. Analytic, p-valent, Ruscheweyh derivative.

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1. INTRODUCTION.

Let \( A(p) \) denote the class of functions of the form

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\})
\]

which are analytic and p-valent in the unit disc \( U = \{z: |z| < 1\} \). For functions \( f_j(z) \) (\( j = 1, 2 \)) defined by

\[
f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k,
\]

we define the convolution product \( f_1 \ast f_2(z) \) of functions \( f_1(z) \) and \( f_2(z) \) by

\[
f_1 \ast f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k.
\]

Using the above convolution product, we define

\[
D^{n+p-1}f(z) = \left( \frac{z^p}{(1-z)^{n+p}} \right)^* f(z) \quad (f(z) \in A(p)),
\]

where \( n \) is any integer greater than \( -p \). We note that

\[
D^{n+p-1}f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!}.
\]

The symbol \( D^{n+p-1} \) when \( p = 1 \) was introduced by Ruscheweyh [8], and the symbol \( D^{n+p-1} \) was introduced by Goel and Sohi [5]. This symbol was named the Ruscheweyh derivative of \( (n+p-1) \)th order by Chen and Owa [4].

It follows from (1.5) that

\[
z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z) \quad (cf. [4] and [5]).
\]

Recently, Chen and Lan ([1] , [2]), Chen, Lee and Owa [3], Chen and Owa [4] and Srivastava, Owa and Pashkouleva [9] have been proved some interesting results of analytic functions involving Ruscheweyh derivatives. In the present paper, we prove an interesting result for functions \( f(z) \in A(p) \) satisfying
In order to prove our main result, we recall here the following lemma:

**LEMMA (Miller [6]; Miller and Mocanu [7]).**

Let $\phi: D \to \mathbb{C}$, $D \subseteq \mathbb{C} \times \mathbb{C}$ (C is the complex plane), and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies the following conditions:

(i) $\phi(u, v)$ is continuous in $D$;
(ii) $(0, 0) \in D$ and $\Re(\phi(0, 0)) > 0$;
(iii) $\Re\{\phi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -\frac{(1 + u_2^2)}{2}$.

Let $q(z) = 1 + q_1z + q_2z^2 + \cdots$ be regular in the unit disc $U$ such that $(q(z), zq'(z)) \in D$ for all $z \in U$. If

$\Re\{\phi(q(z), zq'(z))\} > 0$ \hspace{1cm} (z \in U),

then

$\Re\{q(z)\} > 0$ \hspace{1cm} (z \in U).

Applying the above Lemma, we derive the following:

**THEOREM.** Let the function $f(z)$ be in the class $A(p)$ satisfy

$\Re\left\{\frac{D^{n+p+1}f(z)}{z^p}\right\} > \alpha, \hspace{1cm} (z \in U)$ \hspace{1cm} (2.1)

for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$. Then

$\Re\left\{\sqrt[n]{\frac{D^{n+p}f(z)}{z^p}}\right\} > \beta, \hspace{1cm} (z \in U), \hspace{1cm} (2.2)$

where

$\beta = \frac{1 + \sqrt{1 + 4\alpha(n + p + 1)(n + p + 2)}}{2(n + p + 2)}. \hspace{1cm} (2.3)$

**PROOF.** For $f(z)$ in $A(p)$, we define the function $q(z)$ by

$\sqrt[n]{\frac{D^{n+p}f(z)}{z^p}} = \beta + (1 - \beta)q(z), \hspace{1cm} (2.4)$

where $\beta$ is given by (2.3). Then $q(z)$ is regular in $U$ and $q(z) = 1 + q_1z + q_2z^2 + \cdots$. Taking the derivatives of both sides in (2.4), we have

$\frac{z(D^{n+p}f(z))'}{z^p} - p\frac{(D^{n+p}f(z))}{z^p} = 2(1 - \beta)[\beta + (1 - \beta)q(z)]q'(z). \hspace{1cm} (2.5)$
Since the identity (1.5) implies
\[
(z(D^{n+p}f(z)))' = (n+p+1)D^{n+p+1}f(z) - (n+1)D^{n+p}f(z).
\]
(2.5) becomes
\[
\frac{D^{n+p+1}f(z)}{z^p} = \frac{[\beta + (1-\beta)q(z)]^2 - 2(1-\beta)\beta + (1-\beta)q(z)zq'(z)}{(n+p+1)}.
\]
(2.7)
or
\[
\text{Re} \left\{ \frac{D^{n+p+1}f(z)}{z^p} - \alpha \right\} > \text{Re} \left\{ \frac{[\beta + (1-\beta)q(z)]^2 - 2(1-\beta)\beta + (1-\beta)q(z)zq'(z)}{(n+p+1)} - \alpha \right\} > 0.
\]
(2.8)
Taking \( q(z) = u + u_1 z \) and \( zq'(z) = v + v_1 + iv_2 \), we define the function \( \varphi(u,v) \) by
\[
\varphi(u,v) = \beta + (1-\beta)u + 2(1-\beta)v - \alpha.
\]
(2.9)
Then it follows from (2.9) that
(i) \( \varphi(u,v) \) is continuous in \( D = C \times C \),
(ii) \( (1,0) \in D \) and \( \text{Re} \{ \varphi(1,0) \} = 1 - \alpha > 0 \),
(iii) for all \( (iu_2,v_1) \in D \) such that \( v_1 > -\frac{(1+u_2^2)}{2} \),
\[
\text{Re} \{ \varphi(iu_2,v_1) \} = \beta^2 - (1-\beta)^2 u_2^2 + \frac{2\beta(1-\beta)v_1}{(n+p+1)} - \alpha < 0
\]
for \( 0 < \alpha < 1 \), \( n \in N_0 \), \( n > -p \) and \( \beta \) is given by (2.3). Therefore, the function \( \varphi(u,v) \) satisfies the conditions in the lemma. Thus we have \( \text{Re} \{ q(z) \} > 0 \) \( z \in U \), that is,
\[
\text{Re} \left\{ \frac{D^{n+p}f(z)}{z^p} \right\} > \beta = 1 + \frac{\sqrt{1 + 4\alpha(n+p+1)(n+p+2)}}{2(n+p+2)}.
\]
(2.10)
which completes the proof of the Theorem.

Letting \( \alpha = 0 \), the theorem gives:

COROLLARY 1. Let the function \( f(z) \) be in the class \( A(p) \) satisfy
\[
\text{Re} \left\{ \frac{D^{n+p+1}f(z)}{z^p} \right\} > 0 \quad (z \in U) \quad (2.11)
\]
for \( n \in N_0 \) and \( n > -p \). Then
\[
\text{Re} \left\{ \frac{D^{n+p}f(z)}{z^p} \right\} > -\frac{1}{(n+p+2)} \quad (z \in U). \quad (2.12)
\]
Taking \( n = p \) in the above theorem, we have
COROLLARY 2. Let the function \( f(z) \) be in the class \( A(p) \) satisfy
\[
\Re \left\{ \frac{D^2 f(z)}{z^p} \right\} > \alpha \quad (z \in U)
\]  
for \( 0 \leq \alpha < 1 \) and \( p \in \mathbb{N} \). Then
\[
\Re \left\{ \frac{zf''(z) + (1 - p)f(z)}{z^i} \right\} > 1 + \frac{\sqrt{1 + 24\alpha}}{6}
\]  

REMARK 1. Putting \( p = 1 \) in the above results, we get the results obtained by Chen, Lee and Owa \([3]\). 

REMARK 2. Using the same technique as in the theorem (or putting \( \frac{zf'(z)}{p} \) instead of \( f(z) \) in the theorem), we have the following result:

COROLLARY 3. Let the function \( f(z) \) be in the class \( A(p) \) satisfy
\[
\Re \left\{ \frac{(D^{n+p+1} f(z))'}{pz^{p-1}} \right\} > \alpha \quad (z \in U)
\]  
for \( 0 \leq \alpha < 1 \) and \( n \in \mathbb{N}_0 \). Then
\[
\Re \left\{ \frac{(D^{n+p+1} f(z))'}{pz^{p-1}} \right\} > \beta
\]  

where \( \beta \) is given by (2.3)

REFERENCES

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