NEW CHARACTERIZATIONS FOR HANKEL TRANSFORMABLE SPACES OF ZEMANIAN

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ABSTRACT. In this paper we obtain new characterizations of the Zemanian spaces $H_\mu$ and $H'_\mu$.

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A. H. Zemanian [7, Ch. 5] introduced the space $H_\mu (\mu \in \mathbb{R})$ of functions as follows: a complex valued smooth function $\phi(x)$, $x \in I = (0, \infty)$, is in $H_\mu$ if, and only if, the quantity

$$\gamma_{n,k}^\mu (\phi) = \sup_{x \in I} \left| x^n (x^{-1}D)^k (x^{-\mu-1/2} \phi(x)) \right| < \infty$$

is finite, for every $n, k \in \mathbb{N}$. This space endowed the topology generated by $\{\gamma_{n,k}^\mu\}_{n,k \in \mathbb{N}}$ is a Fréchet space. In the sequel we will refer to the above topology as the usual topology of $H_\mu$. Zemanian introduced the space $H'_\mu$ to extend the Hankel integral transformation defined by

$$\tilde{h}_\mu (x) = \int_0^\infty (xt)^{\mu/2} J_\mu (xt) \phi(t) dt,$$

where $J_\mu$ denotes the Bessel function of the first kind and order $\mu$, to generalized functions. He proved that $h_\mu$ is an automorphism of $H_\mu$ provided that $\mu \geq -\frac{1}{2}$. The generalized Hankel transform $\tilde{h}_\mu$ of $f \in H'_\mu$, the dual space of $H_\mu$, is defined as the transposed of $H'_\mu$ through

$$\langle \tilde{h}_\mu f, \phi \rangle = \langle f, h_\mu \phi \rangle \quad \text{for} \quad \phi \in H_\mu.$$

Thus if $\mu \geq -\frac{1}{2}$, $\tilde{h}_\mu$ is an automorphism of $H'_\mu$ when this space is equipped with the weak* topology or with the strong topology.

In [2] J. J. Betancor and I. Marrero have studied the main topological properties of the spaces $H_\mu$ and $H'_\mu$. Amongst other results, it is established (Theorem 3.3) that the space $H_\mu$, $\mu \geq -\frac{1}{2}$, is constituted by all those complex valued smooth functions $\phi(x)$, $x \in I$, such that

$$\tau_{n,k}^\mu (\phi) = \sup_{x \in I} |x^n N_{\mu+k-1} \cdots N_{\mu} \phi(x)| < \infty$$

for every $n, k \in \mathbb{N}$. Moreover, the system of seminorms $\{\tau_{n,k}^\mu\}_{n,k \in \mathbb{N}}$ generates of $H_\mu$ its usual topology. Moreover in [4] they gave new descriptions for the usual topology of $H_\mu$ through $L_2$-norms.

A. H. Zemanian [7, p. 134] defined the space $O$ formed by all those complex valued smooth functions $v(x)$, $x \in I$, satisfying that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $(1 + x^2)^{n_k} (x^{-1}D)^k v(x)$ is a bounded function on $I$. He proved that $O$ is a space of multiplier of $H_\mu$. Recently J. J. Betancor and I. Marrero [2, Theorems 2.3 and 4.9] have characterized $O$ as the space of multipliers of $H_\mu$ and $H'_\mu$.

In this paper we characterize the smooth complex valued functions in $H_\mu$, $\mu \geq -\frac{1}{2}$, as the ones satisfying

$$Z_n (\phi) = \sup_{x \in I} |x^n \phi(x)| < \infty \quad (1)$$

Moreover in [4] they gave new descriptions for the usual topology of $H_\mu$ through $L_2$-norms.
and
\[ y_n^\mu(\phi) = \sup_{x \in I} |N_{\mu-n-1}\ldots N_\mu \phi(x)| < \infty \] (2)
for every \( n \in \mathbb{N} \). Moreover we prove that the usual topology of \( H^\mu \) can be defined by the family of seminorms \( \{ \Lambda_n, y_n^\mu \}_{n \in \mathbb{N}} \) and a new characterization for the elements of \( H^\mu \) is obtained. In the sequel we will assume that \( \mu \geq -\frac{1}{2} \).

**Proposition 1.** A complex valued smooth function \( \phi(x), \ x \in I \), is in \( H^\mu \) if, and only if, \( \phi \) satisfies (1) and (2) for every \( n \in \mathbb{N} \).

**Proof.** It is clear that if \( \phi \in H^\mu \) then \( \phi \) satisfies (1) and (2) for every \( n \in \mathbb{N} \).

Let now \( \phi \) be a complex valued smooth function defined on \( I \). To see that (1) and (2) \((n \in \mathbb{N})\) are sufficient conditions for \( \phi \) belongs to \( H^\mu \) we proceed by induction. Suppose, as induction hypothesis, that
\[ \sup_{x \in I} |x^{m}N_{\mu+n-1}\ldots N_\mu \phi(x)| < \infty , \quad m \in \mathbb{N} \quad \text{and} \quad \mu \in \mathbb{N} , \quad 0 \leq n < \ell \]
for certain \( \ell \in \mathbb{N}, \ \ell \geq 1 \).

By using partial integration we can obtain
\[
\|x^mN_{\mu-\ell-1}\ldots N_\mu \phi(x)\|^2_2 = \int_0^\infty |x^mN_{\mu-\ell-1}\ldots N_\mu \phi(x)|^2 \, dx \\
= \int_0^\infty x^{2m}N_{\mu-\ell-1}\ldots N_\mu (\phi(x))N_{\mu-\ell-1}\ldots N_\mu (\overline{\phi}(x)) \, dx \\
= \int_0^\infty (Dx^{-1})^\ell(x^{2m+\mu+\ell/2}N_{\mu-\ell-1}\ldots N_\mu (\phi(x)))x^{-\mu-1/2}\overline{\phi}(x) \, dx
\]
for every \( m \in \mathbb{N}, \ \ell < 2m + 2 \), because
\[
[Dx^{-1}]^\ell(x^{2m+\mu+\ell+1/2}N_{\mu-\ell-1}\ldots N_\mu (\phi(x)))(x^{-1}D)^{-\ell-1}(x^{-\mu-1/2}\overline{\phi}(x)) = 0
\] (3)
for each \( i, m \in \mathbb{N}, \ 0 \leq i < \ell < 2m + 2 \). In effect, if \( m, i \in \mathbb{N}, \ 0 \leq i < \ell < 2m + 2 \) then Leibniz's rule leads to
\[
(Dx^{-1})^\ell(x^{2m+\mu+\ell+1/2}N_{\mu-\ell-1}\ldots N_\mu (\phi(x)))(x^{-1}D)^{-\ell-1}(x^{-\mu-1/2}\overline{\phi}(x)) \\
= \sum_{j=0}^\ell a_j x^{2m+2\ell+2m+1/2}(x^{-1}D)^{\ell+i-1}(x^{-\mu-1/2}\overline{\phi}(x)) \\
= \sum_{j=0}^\ell a_j x^{2m+1-i}N_{\mu-\ell+i-j-1}\ldots N_\mu (\phi(x))N_{\mu-\ell+i-2}\ldots N_\mu (\phi(x))
\]
where \( a_j, j \in \mathbb{N}, \ 0 \leq j \leq i \), are suitable real numbers, and by virtue of induction hypothesis (3) follows.

Most straightforward manipulations allow us to write
\[
(Dx^{-1})^\ell(x^{2m+\mu+\ell+1/2}N_{\mu-\ell-1}\ldots N_\mu (\phi(x))x^{-\mu-1/2}\overline{\phi}(x)) = \sum_{j=0}^\ell a_j x^{2m+1-i}N_{\mu+2\ell-j-1}\ldots N_\mu (\phi(x))
\]
with \( m \in \mathbb{N} \) and \( a_j \in \mathbb{R}, \ j \in \mathbb{N}, \ 0 \leq j \leq \ell \).

Hence we can establish
\[
\|x^mN_{\mu-\ell-1}\ldots N_\mu \phi(x)\|^2_2 \leq C_1 \sum_{j=0}^\ell \int_0^\infty |x^{2m-j}\overline{\phi}(x)| |N_{\mu+2\ell-j-1}\ldots N_\mu \phi(x)| \, dx \\
\leq C_2 \sum_{j=0}^\ell \sup_{x \in I} |(1 + x^2)x^{2m-j}\phi(x)| \sup_{x \in I} |N_{\mu+2\ell-j-1}\ldots N_\mu \phi(x)| < \infty ,
\] (4)
provided that \( m \in \mathbb{N}, \ 2m \geq \ell \). Here \( C_{i}, \ i = 1, 2 \), denotes suitable positive constants.

Assume now that \( m \in \mathbb{N}, \ 2m < \ell \). We have

\[
\|x^{m}N_{\mu+\ell-1}...N_{\mu}f(x)\|_{2}^{2} = \left( \int_{0}^{1} + \int_{1}^{\infty} \right) |x^{m}N_{\mu+\ell-1}...N_{\mu}f(x)|^{2} dx \\
\leq \int_{0}^{1} |N_{\mu+\ell-1}...N_{\mu}f(x)|^{2} dx + \int_{1}^{\infty} |x^{\ell}N_{\mu+\ell-1}...N_{\mu}f(x)|^{2} dx.
\]

Therefore, by invoking (4) and the induction hypothesis we infer that

\[
\|x^{m}N_{\mu+\ell-1}...N_{\mu}f(x)\|_{2} < \infty, \quad \text{when} \quad m \in \mathbb{N}, \ 2m \leq \ell.
\]

Thus it is concluded that \( \|x^{m}N_{\mu+\ell-1}...N_{\mu}f(x)\|_{2} < \infty, \ m \in \mathbb{N} \).

Also, for every \( m \in \mathbb{N}, \ m \geq 1, \) and \( x \in I \),

\[
(x^{m}N_{\mu+\ell-1}...N_{\mu}f(x))^{2} = \int_{0}^{\infty} D_{t}^{2}(t^{m}N_{\mu+\ell-1}...N_{\mu}f(t))^{2} dt \\
= \int_{0}^{\infty} 2t^{m}N_{\mu+\ell-1}...N_{\mu}f(t)([m + \mu + \frac{1}{2}] + e(t))^{2} N_{\mu+\ell-1}...N_{\mu}f(t) + t^{m}N_{\mu+\ell}...N_{\mu}f(t)) dt.
\]

Hence if \( m \in \mathbb{N}, \ m \geq 1, \) and \( x \in I \) by using Holder's inequality we can find \( C \geq 0 \) such that

\[
|x^{m}N_{\mu+\ell-1}...N_{\mu}f(x)|^{2} \leq C \left( \|x^{m}N_{\mu+\ell-1}...N_{\mu}f(x)\|_{2}^{2} + \|x^{m-1}N_{\mu+\ell-1}...N_{\mu}f(x)\|_{2}^{2} + \sup_{x \in I} |N_{\mu+\ell-1}...N_{\mu}f(x)| \right)
\]

and then

\[
\sup_{x \in I} |x^{m}N_{\mu+\ell-1}...N_{\mu}f(x)| < \infty, \ m \in \mathbb{N}.
\]

Thus the proof is finished.

The last proposition allows us to define the usual topology of \( H_{\mu} \) through a family of seminorms simpler than \( \{ \gamma_{m,k} \}_{m,k \in \mathbb{N}} \).

**PROPOSITION 2.** The usual topology of \( H_{\mu} \) is defined by the system of seminorms \( \{Z_{n}, y_{m}^{\mu} \}_{n \in \mathbb{N}} \).

**PROOF.** It is clear that the topology generated by \( \{ \gamma_{m,k}^{\mu} \}_{m,k \in \mathbb{N}} \) is finer than the one defined by \( \{Z_{n}, y_{m}^{\mu} \}_{n \in \mathbb{N}} \) on \( H_{\mu} \). Moreover by proceeding in a way similar to A. H. Zemanian [7, Lemma 5.2-2] we can prove that \( H_{\mu} \) endowed with the topology generated by \( \{Z_{n}, y_{m}^{\mu} \}_{n \in \mathbb{N}} \) is a Fréchet space. Hence the desired result is an immediate consequence of the Open Mapping Theorem [6, Corollary 2.12].

We now prove a new characterization for the elements of \( H_{\mu} \), the dual space of \( H_{\mu} \). The procedure employed is analogous to the one used by the author [1] and by J. J. Betancor and I. Marrero [2].

**PROPOSITION 3.** Let \( f \) be a linear functional defined on \( H_{\mu} \). Then \( f \) is in \( H_{\mu}' \) if, and only if, there exist \( r \in \mathbb{N} \) and \( f_{k}, \ g_{k} \in L_{\infty}(0, \infty) \) (the space of essentially bounded functions on \( (0, \infty) \)), \( k \in \mathbb{N}, \ 0 \leq k \leq r \), such that

\[
f = \sum_{k=0}^{r} h_{\mu}^{k}(x^{k}f_{k} + x^{-\mu+1/2}(x^{-1}D)^{k}x^{k+\mu-1/2}g_{k}). \quad (5)
\]

**PROOF.** Let \( f \in H_{\mu}' \). By virtue of a well-known result ([7, Theorem 1.8-1]) there exist \( r \in \mathbb{N} \) and \( C > 0 \) such that

\[
|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq r} \{Z_{k}(\phi), y_{k}^{\mu}(\phi)\}, \quad \phi \in H_{\mu}.
\]

According to [7, Lemma 5.4-1(2), (3) and Theorem 5.4-1] and since \( z^{1/2}J_{\mu}(z) \) is a bounded function on \( I \) for every \( k \in \mathbb{N} \) one has
\[
\sup_{x \in I} |x^k \phi(x)| = \sup_{x \in I} |x^k h_\mu(h_\mu \phi)(x)| \leq C \int_0^\infty |N_{\mu + k - 1} \ldots N_{\mu} (h_\mu \phi)(t)| \, dt \tag{7}
\]

and
\[
\sup_{x \in I} |N_{\mu + k - 1} \ldots N_{\mu} \phi(x)| = \sup_{x \in I} |N_{\mu + k - 1} \ldots N_{\mu} h_\mu(h_\mu \phi)(x)| \leq C \int_0^\infty |t^k (h_\mu \phi)(t)| \, dt \tag{8}
\]

for a suitable \( C > 0 \).

The linear mapping
\[
j : H_\mu \to J H_\mu \subset L_1(0, \infty)^{2r+2}
\]
\[\phi \to (x^k h_\mu \phi, N_{\mu + k - 1} \ldots N_{\mu} h_\mu \phi)_{k=0}^r\]
is one to one because \( h_\mu \) is an automorphism of \( H_\mu \) ([7, Theorem 5.4-1]). Here \( L_1(0, \infty) \) denotes the usual Lebesgue space of order 1.

On the other hand, the inequalities (6), (7) and (8) imply that the linear mapping
\[
L : J H_\mu \subset L_1(0, \infty)^{2r+2} \to \mathbb{C}
\]
\[(x^k h_\mu \phi, N_{\mu + k - 1} \ldots N_{\mu} h_\mu \phi)_{k=0}^r \to \langle f, \phi \rangle\]
is continuous when \( J H_\mu \) is endowed with the topology induced by \( L_1(0, \infty)^{2r+2} \). Hence, by invoking the Hahn-Banach Theorem \( L \) can be extended to \( L_1(0, \infty)^{2r+2} \) as a member of \( (L_1(0, \infty)^{2r+2})' \), the dual space of \( L_1(0, \infty)^{2r+2} \). Since, as it is well known, \( L_1(0, \infty)' = L_\infty(0, \infty) \) there exist \( f_k, g_k \in L_\infty(0, \infty), k \in \mathbb{N}, 0 \leq k \leq r \), such that
\[
\langle f, \phi \rangle = \sum_{k=0}^r \langle (f_k, x^k h_\mu \phi) + (g_k, x^{k+1/2} (x^{-1} D)^k (x^{-1/2} \phi)) \rangle, \quad \phi \in H_\mu.
\]

Therefore
\[
f = \sum_{k=0}^r h_\mu (x^k f_k + (-1)^k x^{k+1/2} (x^{-1} D)^k x^{k+1/2} g_k).
\]

Thus the proof of necessity if finished.

Conversely, if \( f \) is a linear functional defined on \( H_\mu \) by (5) for certain \( r \in \mathbb{N} \) and \( f_k, g_k \in L_\infty(0, \infty), k \in \mathbb{N}, 0 \leq k \leq r \), then
\[
|\langle f, \phi \rangle| \leq C \sum_{k=0}^r \|f_k\|_\infty \sup_{x \in I} |(1 + x^2) x^k (h_\mu \phi)(x)| + \|g_k\|_\infty \sup_{x \in I} |(1 + x^2) N_{\mu + k - 1} \ldots N_{\mu} (h_\mu \phi)(x)|
\]

for \( \phi \in H_\mu \), where \( \|\cdot\|_\infty \) denotes the usual norm in \( L_\infty(0, \infty) \). Hence, according to [7, Theorem 5.4-1] and [2, Theorem 3.3], \( f \) is in \( H_\mu' \).

REFERENCES

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