ON A CLASS OF EXACT LOCALLY CONFORMAL COSYMPLECTIC MANIFOLDS

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ABSTRACT. An almost cosymplectic manifold M is a $(2m + 1)$ -dimensional oriented Riemannian manifold endowed with a 2-form Ω of rank $2m$, a 1-form η such that $\Omega^m \wedge \eta \neq 0$ and a vector field ξ satisfying $i_\xi \Omega = 0$ and $\eta(\xi) = 1$. Particular cases were considered in [3] and [6].

Let (M, g) be an odd dimensional oriented Riemannian manifold carrying a globally defined vector field T such that the Riemannian connection is parallel with respect to T . It is shown that in this case M is a hyperbolic space form endowed with an exact locally conformal cosymplectic structure. Moreover T defines an infinitesimal homothety of the connection forms and a relative infinitesimal conformal transformation of the curvature forms.

The existence of a structure conformal vector field C on M is proved and their properties are investigated. In the last section, we study the geometry of the tangent bundle of an exact locally conformal cosymplectic manifold.

KEY WORDS AND PHRASES: Locally conformal cosymplectic manifold, T -parallel connection, infinitesimal homothety, infinitesimal conformal transformation, Hamiltonian vector field, tangent bundle, Liouville vector field, complete lift, mechanical system.

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1. INTRODUCTION

In the last decade a series of papers have been devoted to almost cosymplectic manifolds $M(\Omega, \eta, \xi, g)$. As is well known, an almost cosymplectic manifold M is an odd dimensional (say $2m + 1$) oriented manifold, where the triple (Ω, η, ξ) of tensor fields is

- i) a 2-form Ω of rank $2m$
- ii) a 1-form η such that $\Omega^m \wedge \eta \neq 0$
- iii) a vector field (called the Reeb vector field) such that $i_\xi \Omega = 0$ and $\eta(\xi) = 1$.

One has the following more studied cases:

- 1° Ω and η are both closed forms. Then M is called a cosymplectic manifold.
- 2° $d\eta = 0$, $d\Omega = 2\eta \wedge \Omega$. Then M is called a Kenmotsu manifold.
- 3° $d\eta = \omega \wedge \eta$, $d\Omega = 2\omega \wedge \Omega$. Then M is called a locally conformal cosymplectic manifold (see [3], [16]). In this case ω and its dual vector $T = b^{-1}(\omega)$ with respect to g is called the Lee form (or characteristic form) and Lee vector field respectively.

In the present paper we consider an almost cosymplectic manifold $M(\Omega, \eta, \xi, g)$ carrying a globally defined vector field T whose dual form $b(T)$ is denoted by ω .

Next denote by $0 = \text{vect}\{e_A; A = 0, 1, \dots, 2m\}$ an orthonormal vector basis on M and by $\left\{\theta_B^A\right\}$ the associated connection forms. If the connection forms satisfy

$$\theta_B^A = \langle T, e_B \wedge e_A \rangle; \quad \wedge \text{ is the wedge product,}$$

then one has

$$\nabla_T e_A = 0$$

Therefore we agree to say that M is structured by a T -parallel connection. In this condition the following significative fact emerges: the almost cosymplectic structure $1 \times Sp(2m, \mathbf{R})$ of M moves to an exact locally conformal cosymplectic structure $1 \times Sp(2m, \mathbf{R})$ (abbreviated exact L.C.C.), having T (resp. $\omega = -df/f$) as Lee vector field (resp. Lee form).

Moreover any such a manifold M is a space form of curvature $-2c$ and f is the energy function corresponding to a Hamiltonian vector field associated with T (in the sense of [3]). If θ (resp. Θ) represents the indexless (or generic) connection forms (resp. curvature forms) of M , then T defines an infinitesimal homothety of θ , i.e. $L_T \theta = 2c\theta$, and a relative infinitesimal T conformal transformation of Θ and Ω , i.e.

$$d(L_T \Theta) = 2c\omega \wedge \Theta, \quad d(L_T \Omega) = 2c\omega \wedge \Omega.$$

In Section 3 the existence of a structure conformal vector field C on M is proved, i.e.

$$\nabla_Z C = \lambda Z + g(Z, T)C - g(Z, C)T; \quad \lambda \in C^\infty M, \quad Z \in \Gamma(TM).$$

Moreover C is a divergence conformal vector field, i.e. $\text{grad}(\text{div } C)$ is a concurrent vector field and it defines an infinitesimal conformal transformation of:

- i) the conformal cosymplectic form Ω , i.e. $L_C \Omega = \rho \Omega$, $\rho = 2\lambda$;
- ii) the dual forms ω^A , i.e. $L_C \omega^A = \frac{\rho}{2} \omega^A$;
- iii) the curvature forms Θ_b^A , i.e. $L_C \Theta_b^A = \rho \Theta_b^A$;
- iv) all the $(2q+1)$ -forms $\alpha_q = b(C) \wedge \Omega^q$, i.e. $L_C \alpha_q = (1+q)\rho \alpha_q$;
- v) all the functions $g(C, Z)$, i.e. $L_C g(C, Z) = \rho g(C, Z)$, $Z \in \Gamma(TM)$.

In the last section, we discuss some properties of the tangent bundle manifold TM having as basis the exact (L.C.C.)-manifold M . Denote by V, γ and ν the Liouville vector field ([13]), the Liouville 1-form and the Liouville function respectively, on TM .

The following properties are proved:

- i) the complete lift Ω^c of Ω is a d^ω -exact 2-form (d^ω is the cohomological operator [11]) and is homogeneous of class 1, i.e.

$$L_V \Omega^c = \Omega^c;$$

- ii) γ satisfies $d^\omega \gamma = \psi$ and ψ is a Finslerian form, i.e.

$$L_V \psi = \psi, \quad i_V \psi = 0$$

(i_V denotes the vertical differentiation operator [11]);

- iii) the vertical lift T^ν of T defines an infinitesimal automorphism of ψ , i.e. $LT^\nu \psi = 0$;
- iv) the function $r = f\nu$ and the 2-form $f\psi$ define a regular mechanical system $\mathcal{M}$ ([13]) having r as kinetic energy and $f\psi$ as canonical symplectic (exact) form.

1. PRELIMINARIES

Let (M, g) be a Riemannian C^∞ -manifold and let ∇ be the covariant differential operator with respect to the metric tensor g . Assume that M is oriented and ∇ is a Levi-Civita connection. Let $\Gamma(TM) = \chi(M)$ and $b: TM \rightarrow T^*M$ be the set of sections of the tangent bundle TM and the musical isomorphism ([18]) defined by g , respectively. Following [18] we set

$$\Lambda^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM)$$

and notice that elements of $A^q(M, TM)$ are vector valued q -forms ($q \leq \dim M$).

Denote by $d^\nabla: A^q(M, TM) \rightarrow A^{q+1}(M, TM)$ the exterior covariant derivative operator with respect to ∇ . It should be noticed that generally $d^\nabla = d^\nabla \circ d^\nabla \neq 0$ unlike $d^2 = d \circ d = 0$. If $p \in M$, then the vector valued 1-form $dp \in A^1(M, TM)$ is the canonical vector valued 1-form of M ([5]) and since ∇ is symmetric one has $d^\nabla(dp) = 0$. The operator

$$d^\omega = d + e(\omega) \quad (1.1)$$

acting on ΛM , where $e(\omega)$ means the exterior product by the closed 1-form ω , is called the cohomological operator ([11]). One has

$$d^\omega \circ d^\omega = 0. \quad (1.2)$$

Any form $u \in \Lambda M$ such that $d^\omega u = 0$ is said to be d^ω -closed and if ω is an exact form, then u is said to be a d^ω -exact form. Any vector field $Z \in \Gamma(TM)$ such that

$$d^\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM) \quad (1.3)$$

for some 1-form π , is said to be an exterior concurrent vector field ([17]). The form π which is called the concurrence form is given by

$$\pi = \lambda b(Z); \quad \lambda \in C^\infty M. \quad (1.4)$$

A non flat manifold of dimension $m > 2$ is an elliptic or hyperbolic space-form if and only if every vector field on M is an exterior concurrent one ([17]). On the tangent bundle manifold TM , d_v and i_v define the vertical differentiation and the vertical derivation operators respectively ([7]). d_v is an anti-derivation of degree 1 on $\Lambda(TM)$ and i_v is a derivation of degree 0 on $\nabla(TM)$.

In an n -dimensional Riemannian manifold M , denote by

$$O = \text{vect}\{e_A; A = 1, \dots, n\}$$

a local field of orthonormal frames and let

$$O^* = \text{covect}\{\omega^A; A = 1, \dots, n\}$$

be its associated coframe.

The soldering form dp is expressed by

$$dp = \omega^A \otimes e_A \quad (1.5)$$

and E. Cartan's structure equations written indexless manner are

$$\nabla e = \theta \otimes e \quad (1.6)$$

$$d\omega = -\theta \wedge \omega \quad (1.7)$$

$$d\theta = -\theta \wedge \theta + \Theta \quad (1.8)$$

Any vector field T such that

$$\nabla T = s dp + u \otimes T, \quad u \in \Lambda^1 M \quad (1.9)$$

is called a torse forming (K. Yano [20]). If $du = 0$, then T is a closed torse forming, which implies that T is an exterior concurrent vector field, and if $u = 0$, then T is a concurrent vector field ([22]).

Let now W be any conformal vector field on M (i.e. the conformal version of Killing's equations). As is well known, W satisfies

$$L_W g = \rho g \quad \text{or} \quad g(\nabla_Z W, Z') + g(\nabla_{Z'} W, Z) = \rho g(Z, Z') \quad (1.10)$$

where the conformal scalar ρ is defined by

$$\rho = \frac{2}{n}(\text{div} W). \quad (1.11)$$

We recall some basic formulas which we shall use in the following sections.

$$L_W b(Z) = \rho b(Z) + b[W, Z] \quad (\text{Orsted lemma}) \quad (1.12)$$

$$L_W K = (n-1)\Delta\rho - K\rho \quad (1.13)$$

$$2L_W S(Z, Z') = (\Delta)\rho g(Z, Z') - (n-2) \text{ (Hess } \nabla^p)(Z, Z'). \quad (1.14)$$

In the above equations L_W , K , Δ and S denote the *Lie* derivative with respect to W , the scalar curvature of M , the Laplacian and the Ricci tensor field of ∇ , respectively. One has

$$(\text{Hess}_\nabla \rho)(Z, Z') = g(Z, H_\rho Z'), \quad H_\rho Z' = \nabla_{Z'}(\text{grad } \rho)$$

(see also [2]).

2. EXACT LOCALLY CONFORMAL COSYMPLECTIC MANIFOLDS

Let (M, g) be a $(2m+1)$ -dimensional oriented Riemannian C^∞ -manifold and let $T = \sum_{A=0}^{2m} t^A e_A$ and $\omega = b(T)$ be a globally defined vector field on M and its dual form respectively.

Denote by $\mathbf{O} = \text{vect}\{e_A; A = 0, 1, \dots, 2m\}$ (resp. θ_B^A) a local field of orthonormal frames on M (resp. the associated connection forms). Recall that the vectorial wedge product $\wedge$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y; \quad Z \in \Gamma(TM)$$

i.e.

$$X \wedge Y = b(Y) \otimes X - b(X) \otimes Y.$$

Assume now that all the connection forms θ satisfy

$$\theta_B^A = \langle T, e_B \wedge e_A \rangle. \quad (2.1)$$

Then by the structure equations (1.6), it follows at once

$$\theta_B^A = t^B \omega^A - t^A \omega^B. \quad (2.2)$$

It should be noticed that if θ satisfy (2.2) one has $\theta(T) = 0$ and the above equation shows that all the connection forms θ are relations of integral invariance for the vector field T (in the sense of A. Lichnerowicz [14]).

Next by the structure equations (1.6) and by (2.2) one obtains

$$\nabla e_A = t^A dp - \omega^A \otimes T \quad (2.3)$$

and the above equation implies

$$\nabla_T e_A = 0. \quad (2.4)$$

From (2.4) the following significative fact emerges: all the vectors of the $\mathbf{O}$ -basis are T -parallel. Therefore we agree to say that the Riemannian manifold under consideration is structured by a T -parallel connection (abr. T.P.).

Further again by (2.2) one derives by the structure equations (1.7)

$$d\omega^A = \omega \wedge \omega^A; \quad \omega = b(T) = t^A \omega^A \quad (2.5)$$

which by a simple argument implies that the dual form ω of T is closed, i.e.

$$d\omega = 0. \quad (2.6)$$

Thus in terms of d^ω -cohomology, (2.5) may be written as

$$d^\omega \omega^A = 0 \quad (2.7)$$

and $\mathbf{O}^* = \{\omega^A\}$ is defined as a d^ω -closed covector basis.

Now for reasons which will soon appear, we set

$$\omega^0 = \eta, \quad e_0 = \xi \quad (2.8)$$

and consider on M the globally defined 2-form Ω of rank $2m$ given by

$$\Omega = \Sigma \omega^a \wedge \omega^{a*}; \quad a = 1, \dots, m; \quad a^* = a + m. \quad (2.9)$$

Then since $\Omega^m \wedge \eta \neq 0$, $i_\xi \Omega = 0$, one may say that the triple (Ω, η, ξ) defines an almost cosymplectic structure $1 \times Sp(2m, \mathbf{R})$ having ξ as Reeb's vector field.

Next taking the exterior differential of Ω a short calculation gives with the help of (2.5)

$$d\Omega = 2\omega \wedge \Omega \Leftrightarrow d^{-2m}\Omega = 0 \quad (2.10)$$

and by (2.5) we may write

$$d\eta = \omega \wedge \eta \Leftrightarrow d^{-m}\eta = 0. \quad (2.11)$$

We conclude that any odd dimensional Riemannian manifold M structured by a T -parallel connection is endowed with a locally conformal cosymplectic structure $1 \times CSp(2n, \mathbf{R})$ (abr. L.C.C.). We notice that the vector field T (resp. the 1-form $\omega = b(T)$) is the Lee vector field (resp. the Lee form) of this structure.

Moreover since $\omega = t^A \omega^A$, then by a simple argument it follows on behalf of (2.5) that one may set

$$dt^A = f\omega^A; \quad f \in C^\infty M \quad (2.12)$$

which by exterior differentiation gives instantly

$$\omega = -df/f. \quad (2.13)$$

Therefore since ω is an exact form, it follows on behalf of a known terminology, that the manifold M under consideration is an exact (L.C.C.)-manifold. We agree to call f the distinguished scalar field associated with the exact (L.C.C.)-structure.

Now taking the covariant differential of T one finds by (2.3) and (2.12)

$$\nabla T = (f + 2l)dp - \omega \otimes T \quad (2.14)$$

where we have set

$$g(T, T) = 2l. \quad (2.15)$$

Using (2.12) and (2.15), we have

$$dl = f\omega \Rightarrow l + f = c = \text{const} \neq 0 \quad (2.16)$$

and (2.14) becomes

$$\nabla T = (l + c)dp - \omega \otimes T. \quad (2.17)$$

Hence, by (1.9) and (2.6) T is a closed torse forming and consequently an exterior concurrent (abr. E.C.)-vector field.

Operating now on ∇e_A and ∇T by the exterior covariant derivative operator d^∇ , one gets by (2.12) and (2.16)

$$d^\nabla(\nabla e_A) = \nabla^2 e_A = 2c\omega^A \wedge dp \quad (2.18)$$

$$d^\nabla(\nabla T) = \nabla^2 T = 2c\omega \wedge dp. \quad (2.19)$$

From the above equations it is seen that any vector field Z on M is E.C. with constant conformal scalar $2c$. Therefore on behalf of the general properties of E.C.-vector fields ([17]), we may state the following striking property: the exact L.C.C.-manifold $M(\Omega, \eta, \xi)$ under discussion is a space-form of curvature $-2c$.

As a consequence, it follows that the curvature forms Θ are expressed by

$$\Theta_B^A = -2c\omega^A \wedge \omega^B \quad (2.20)$$

Next taking the exterior differential of the forms Θ , one quickly finds by

$$d\Theta_B^A = 2\omega \wedge \Theta_B^A \Leftrightarrow d^{-2m}\Theta_B^A = 0 \quad (2.21)$$

which shows that all the curvature forms Θ are $d^{-2\omega}$ -exact.

On the other hand taking the Lie derivatives of the covectors ω^A of $\mathbf{O}^*$ one derives by (2.12) and (2.16)

$$L_I \omega^A = (l + c) \omega^A - t^A \omega. \quad (2.22)$$

Therefore since L_I satisfies Leibniz rule one deduces by (2.20)

$$L_I \Theta_B^A = 2(l + c) \Theta_B^A + 2c \Theta_B^A \wedge \omega \quad (2.23)$$

Similarly, we obtain

$$d\Theta_B^A = 2f\omega^B \wedge \omega^A + \omega \wedge \Theta_B^A \quad (2.24)$$

Clearly by (2.12) one has $L_I t^A = f t^A$ and with the help of (2.22) we deduce

$$L_I \Theta_B^A = 2c \Theta_B^A. \quad (2.25)$$

Accordingly by the above equations we may say that the Lie vector field T defines on infinitesimal homothety of all the connection forms θ .

Taking now the exterior differential of the equations (2.23), a standard calculation gives

$$d(L_I \Theta_B^A) = 8c \omega \wedge \Theta_B^A \quad (2.26)$$

which proves that T defines a relative infinitesimal conformal transformation ([19]) of the curvature forms.

let $\mu : TM \rightarrow T^*M$, $\mu(Z) = i_Z \Omega$ be the bundle isomorphism defined by Ω and set $\bar{\omega} = \mu(T)$, i.e.

$$\bar{\omega} = i_T \Omega = \sum_{a=1}^m (t^a \omega^{a*} - t^{a*} \omega^a) \quad (2.27)$$

for the dual form of T with respect to Ω . By (2.5) and (2.12) an easy calculation gives

$$d\bar{\omega} = 2f\Omega + \omega \wedge \bar{\omega} \quad (2.28)$$

and by (2.10) and (2.13) one gets

$$L_T \Omega = 2(l + c)\Omega + \bar{\omega} \wedge \omega \quad (2.29)$$

and consequently by (2.28) it follows

$$d(L_T \Omega) = 2c \omega \wedge \Omega. \quad (2.30)$$

Hence as for the curvature forms Θ , T defines a relative conformal transformation of the structure 2-form Ω .

Consider now the vector valued 1-form

$$F = \omega^a \otimes e_{a*} - \omega^{a*} \otimes e_a \in A^1(M, TM). \quad (2.31)$$

If Z is any vector field, a simple calculation gives

$$\langle F, Z \rangle = Z^a e_{a*} - Z^{a*} e_a = \bar{Z} \quad (2.32)$$

which implies

$$g(Z, Z') + g(Z, \bar{Z}') = 0, \quad Z, Z' \in \Gamma(TM) \quad (2.33)$$

and $\langle F, dp \rangle = 2\Omega$.

On the other hand since $\bar{\omega}(T) = 0$ one gets by (2.27)

$$L_T \bar{\omega} = 2c \bar{\omega} \quad (2.34)$$

that is T defines an infinitesimal homothety of $\bar{\omega} = (\mu \circ b)T$.

Next by (2.12) and (2.13) one easily gets

$$i_T \Omega = \frac{df}{f} - \frac{1}{f} \xi(f) \eta. \quad (2.35)$$

Therefore by reference to [3] one may call $\bar{T}$ the cosymplectic Hamiltonian vector field of M and the distinguished scalar f turns out to be the energy function corresponding to $\bar{T}$.

Moreover by (2.35) one derives

$$L_{\bar{T}} \Omega = \eta(\bar{T}) \eta \wedge \omega \Rightarrow d(L_{\bar{T}} \Omega) = 0 \quad (2.36)$$

which shows that $\bar{T}$ defines a relative infinitesimal automorphism (R. Abraham [1]) of Ω .

Summing up, we state the following

THEOREM. Let M be a $(2m+1)$ -dimensional Riemannian manifold and let T be a globally defined vector field on M . If M is structured by a T -parallel connection, then M is endowed with an exact locally conformal cosymplectic structure $1 \times CSp(2m, \mathbf{R})$, having T (resp. $\omega = b(T)$) as Lee vector (resp. Lee form) and any such an M is a space-form of curvature $-2c$.

Moreover one has the following properties:

- i) T defines an infinitesimal homothety of the connection forms θ and of the 1-form $\mu(T)$, i.e.

$$L_T \theta = 2c \theta, \quad L_T \mu(T) = 2c \mu(T)$$

- ii) T defines a relative infinitesimal conformal transformation of the curvature forms Θ and of the structure 2-form Ω , i.e.

$$d(L_T \Theta) = 8c \omega \wedge \Theta, \quad d(L_T \Omega) = 2c \omega \wedge \Omega$$

- iii) the vector field $\bar{T} = (b^{-1} \circ \mu) T$ (resp. f) is the cosymplectic Hamiltonian associated with the $1 \times CSp(2m, \mathbf{R})$ -structure of M (resp. its corresponding energy function) and $\bar{T}$ defines a relative infinitesimal automorphism of Ω .

Let now $\Phi: M \rightarrow \bar{M}$ be a conformal diffeomorphism (abr. C.D.) that is

$$\Phi: g \rightarrow e^{2\sigma} g = \bar{g}; \quad \sigma \in C^\infty M.$$

One also say that g and $\bar{g}$ are conformally equivalent metrics and setting $e^{2\sigma} = v^2$, we agree to call the function v the argument of the C.D.

As is shown one has for $Z, Z' \in \Gamma(TM)$

$$\hat{\nabla} Z = \nabla Z + b(\text{grad } \sigma) \otimes Z - b(Z) \otimes \text{grad } \sigma + g(Z, \text{grad } \sigma) dp \quad (2.37)$$

or equivalently

$$\hat{\nabla}_Z Z = \nabla_Z Z + Z'(\sigma)Z + Z(\sigma)Z' - g(Z, Z') \text{grad } \sigma \quad (2.38)$$

and if K and $\bar{K}$ denote the scalar curvature of M and $\bar{M}$ respectively then one has ([8])

$$\bar{K} = e^{-2\sigma} \{K + 2(n-1)(n-2) \|\text{grad } \sigma\|^2\} \quad (2.39)$$

($n = \dim M$).

If M is an exact (L.C.C.)-manifold, its Ricci tensor field S satisfies

$$S(Z, Z') = -4mc g(Z, Z'); \quad Z, Z' \in \Gamma(TM) \quad (2.40)$$

and the scalar curvature K is given by

$$K = -4m(2m+1)c. \quad (2.41)$$

Perform now a conformal transformation of M having as argument e^σ the energy function f . It is obvious that

$$(2.42) d\sigma = df/f = -\omega. \quad (2.42)$$

Then we have $\text{grad } \sigma = -T$, which implies

$$\Delta\sigma = \operatorname{div} T = (2m+1)c + (2m-1)l. \quad (2.43)$$

Hence by (2.41) and (2.43) we derive at once from (2.39), $\tilde{K} = 0$, that is $\tilde{M}$ is a flat manifold. We notice that this fact is in accordance with the known

PROPOSITION. A Riemannian manifold of constant curvature is conformally flat, provided $n \geq 3$.

Using (2.37) one may prove that all vectors $\tilde{e}_A$ are parallel (the connection forms $\tilde{\theta}_B^A$ vanish, i.e. $\tilde{\nabla}$ is a flat connection). Thus we have

PROPOSITION. If M is an exact (L.C.C.)-manifold with metric tensor g and energy function f , then the metric $f^2 g$ is flat.

3. STRUCTURE CONFORMAL VECTOR FIELDS ON AN EXACT (L.C.C.)-MANIFOLD

In consequence of some conformal properties induced by the T -parallel connection which structures $M(\Omega, \eta, \xi, g)$ we are naturally led to see if the manifold M under consideration carries a structure conformal vector field C in the sense of [6], [15]. Therefore the covariant differential of C is expressed by

$$\nabla C = \lambda dp + C \wedge T = \lambda dp + \omega \otimes C - \alpha \otimes T; \quad \lambda \in C^\infty M, \quad \alpha = b(C). \quad (3.1)$$

Put

$$C = C^A e_A \Rightarrow b(C) = \alpha = C^A \omega^A \quad (3.2)$$

and $s = g(C, T)$. Then by (2.3) and (3.1) one quickly gets

$$dC^A = (\lambda - s)\omega^A + C^A \omega \quad (3.3)$$

$$d\alpha = 2\omega \wedge \alpha \Rightarrow d^{-2\omega}\alpha = 0. \quad (3.4)$$

Next since $ds = \langle \nabla C, T \rangle + \langle \nabla T, C \rangle$, a short calculation gives

$$ds = \lambda \omega - (l - c)\alpha \quad (3.5)$$

$$ds = d\lambda. \quad (3.6)$$

By (3.4), (3.5) and (3.6) it is seen that the existence of C is assured by an exterior differential system Σ whose characteristic numbers are

$$r = 3, \quad s_0 = 2, \quad s_1 = 1.$$

Then Σ is in involution in the sense of E. Cartan (i.e. $r = s_c + s_l$). Accordingly one may say that the existence of C depends on 2 arbitrary functions of one argument (E. Cartan's test). The conformal scalar ρ associated with $C(L_C g = \rho g)$ is given by

$$\rho = 2\lambda. \quad (3.7)$$

By a short calculation one has

$$[C, T] = -\lambda T - (l - c)C; \quad [\quad]: \text{Lie bracket} \quad (3.8)$$

and from (3.5) it follows

$$L_C \omega = ds = \lambda \omega - (l - c)\alpha. \quad (3.9)$$

This equation matches by Orsted's lemma (1.12) the expression of $[C, T]$.

On the other hand since C is necessarily an E. C. vector field (M is a space-form), then operating (3.1) by d^∇ and taking account of (3.4) and (3.5), one derives

$$d^\nabla(\nabla C) = \nabla^2 C = 2c\alpha \wedge dp. \quad (3.10)$$

The above equation is coherent with the properties obtained in Section 2.

Setting now

$$\bar{\alpha} = \iota_C \Omega = \Sigma(C^a \omega^{a*} - C^{a*} \omega^a) \quad (3.11)$$

one gets by (3.4) and (2.5)

$$d\bar{\alpha} = 2(\lambda - s)\Omega + 2\omega \wedge \bar{\alpha} \quad (3.12)$$

and one follows

$$L_C \Omega = \rho \Omega . \quad (3.13)$$

Hence (3.13) reveals that C defines an infinitesimal conformal transformation (abr. I.C.T.) of the conformal cosymplectic form Ω .

By similar methods, one gets by (2.5), (2.24), (2.20) and (2.21)

$$L_C \omega^A = \frac{\rho}{2} \omega^A, \quad L_C \theta_B^A = \frac{\rho}{2} \theta_B^A, \quad L_C \Theta_B^A = \rho \Theta_B^A . \quad (3.14)$$

Therefore one may say that C defines an I.C.T. of the exact (L.C.C.)-structure of M .

Moreover let L be the operator of type (1.1) on forms defined by S . Goldberg ([8]), that is $Lu = u \wedge \Omega; u \in \wedge^1 M$, and consider on M the $(2q + 1)$ -forms

$$L^q \alpha = \alpha_q = \alpha \wedge \Omega^q . \quad (3.15)$$

Since by Orsted's lemma one has

$$L_C \alpha = \rho \alpha \quad (3.16)$$

then by (3.13) and a standard calculation one derives

$$L_C \alpha_q = (q + 1)\rho \alpha_q . \quad (3.17)$$

Hence C defines an (I.C.T.) of all the $(2q + 1)$ -forms α_q .

Next since C is a conformal vector field, then as is known (see (1.11)) one has

$$\operatorname{div} C = (\rho/2)(2m + 1) \quad (3.18)$$

and since $\rho = 2\lambda$ it follows by (3.5) and (3.6) that

$$\operatorname{grad} \rho = \rho T + 2(c - l)C . \quad (3.19)$$

Further by (2.16) and taking account of (2.14) and (3.1) it is easily deduced

$$\nabla \operatorname{grad} \rho = 2c \rho d\rho . \quad (3.20)$$

Thus one may state the following relevant property: the gradient of the associated scalar ρ of C is a concurrent vector field (K. Yano and B. Y. Chen [22]). We agree to call a conformal vector field such that the gradient of its conformal scalar ρ is a concurrent vector field, a divergence conformal vector field. Such a situation occurs also when studying conformal vector fields on Lorentzian P.S. manifolds (see I. Mihai and R. Rosca [15]).

On the other hand from (2.14) one derives

$$\operatorname{div} T = (2m - 1)l + (2m + 1)c \quad (3.21)$$

and since $\operatorname{div} C = (2m + 1)\lambda$, one gets on behalf of (3.20)

$$\Delta \rho = -\operatorname{div}(\operatorname{grad} \rho) = -2(2m + 1)c\rho \quad (3.22)$$

which shows that ρ is an eigenfunction of Δ .

C being an E.C. vector field satisfying (3.10), one has ([17])

$$S(C, Z) = -4mc g(C, Z), \quad Z \in \Gamma(TM) \quad (3.23)$$

where S denotes the Ricci tensor field of ∇ .

Now making use of (1.14) and carrying out the calculations, one finds by (3.19) and (3.22)

$$L_C g(C, Z) = \rho g(C, Z) . \quad (3.24)$$

Hence the vector field C defines an I.C.T. of all the functions $g(C, Z)$, where $Z \in \Gamma(TM)$.

Concluding, we have proved the following

THEOREM. Let M be the exact (L.C.C.) manifold defined in Section 2 and C a structure conformal vector field on M (which existence is proved), i.e.

$$\nabla C = \frac{\rho}{2} dp + C \wedge T; \quad L_C g = \rho g$$

Then C is a divergence conformal vector field (i.e. $\text{grad}(\text{div } C)$ is a concurrent vector field) and it defines the following infinitesimal conformal transformations

$$L_C \Omega = \rho \Omega, \quad L_C \omega^i = \frac{\rho}{2} \omega^i, \quad L_C \theta_h^i = \frac{\rho}{2} \theta_h^i$$

$$L_C \Theta_h^i = \rho \Theta_h^i, \quad L_C \alpha_q = (1 + q) \rho \alpha_q, \quad L_C g(C, Z) = \rho g(C, Z) (Z \in \Gamma(TM))$$

where Ω , ω^i , θ_h^i , Θ_h^i and $\alpha_q = b(C) \wedge \Omega^q$ are the conformal symplectic 2-form, the dual forms, the connection forms, the curvature forms and the $(2q + 1)$ -forms defined by the $(1, 1)$ -operator L , respectively on M .

4. GEOMETRY OF THE TANGENT BUNDLE OF AN EXACT (L.C.C.)-MANIFOLD

Let now TM be the tangent bundle manifold having the exact (L.C.C.)-manifold M discussed in Section 2 as a basis.

Denote by $V(v^A) (A = 0, 1, \dots, 2m)$ the Liouville vector field (or the canonical vector field [7]). Accordingly we may consider the set $B^* = \{\omega^A, dv^A\}$ as an adapted cobasis in TM . Following Godbillon ([7]) we denote by d_v and i_v the vertical differentiation and the vertical derivative operators with respect to B^* , respectively (d_v is an antiderivation of degree 1 on $\Lambda(TM)$ and i_v is a derivation of degree 0 on $\Lambda(TM)$). Let $T_s^r M$ be the set of all tensor fields of type (r, s) on M .

In general as is known ([23]) the vertical and complete lifts are linear mappings of $T_s^r M$ into $T_s^r(TM)$ and one has

$$(T_1 \otimes T_2)^c = T_1^v \otimes T_2^c + T_1^c \otimes T_2^v. \quad (4.1)$$

In the case under discussion we may define the complete lift Ω^c of the structure 2-form Ω of M by the 2-form of rank $4m$ on TM

$$\Omega^c = \Sigma(dv^a \wedge \omega^{a^*} + \omega^a \wedge dv^{a^*}), \quad a = 1, \dots, m; \quad a^* = a + m. \quad (4.2)$$

On the other hand since the Liouville vector field V is expressed by

$$V = \Sigma v^A \frac{\partial}{\partial v^A} \quad (4.3)$$

then as is known the basic 1-form

$$\gamma = \Sigma v^A \omega^A \quad (4.4)$$

is called the Liouville form (see also [13]).

Taking now the exterior differential of Ω^c one finds by (2.5)

$$d\Omega^c = \omega \wedge \Omega^c \Leftrightarrow d^{-\omega} \Omega^c = 0 \quad (4.5)$$

which shows that Ω^c is similarly as Ω a $d^{-\omega}$ -exact form. We recall that in general conformal properties are not preserved by complete lifts ([23]).

One has

$$i_v \Omega^c = \Sigma(v^a \omega^{a^*} - v^{a^*} \omega^a) \quad (4.6)$$

which implies $\omega(V) = 0$ and so by (4.5) and (4.6) one gets

$$L_v \Omega^c = \Omega^c. \quad (4.7)$$

Accordingly on behalf of a known definition ([13]), the above equation shows that Ω^c is of class 1, a homogeneous form on TM . Taking now the exterior differential of the Liouville form γ defined by (4.4), one gets at once by (2.5)

$$d\gamma = \omega \wedge \gamma + \psi \Leftrightarrow d^{-\omega} \gamma = \psi \quad (4.8)$$

where we have set

$$\psi = \sum d v^A \wedge \omega^A . \quad (4.9)$$

From (4.8) and (1.2) one obtains instantly

$$d^{m-1} \psi = 0 \Leftrightarrow d \psi = \omega \wedge \psi . \quad (4.10)$$

Since clearly the 2-form ψ is of maximal rank, we agree to call ψ the canonical conformal symplectic form of M . Noticing that one has

$$\iota_V \psi = \gamma , \quad \omega(V) = 0 \quad (4.11)$$

which implies

$$L_V \psi = \psi . \quad (4.12)$$

Hence ψ is as Ω^2 a homogeneous of class 1, 2-form.

Next making use of the vertical operator i_v defined by $i_v \lambda = 0$, $i_v d v^A = \omega^A$, $i_v \omega^A = 0$ ($\lambda \in C^{\infty} M$) one quickly finds by (4.9)

$$i_v \psi = 0 \quad (4.13)$$

and the above equation together with (4.12) proves that ψ is a Finslerian form ([7]).

We recall that the vertical lift Z^v ([23]) of a vector field $Z \in \Gamma(TM)$ with components Z^A in M , has as components

$$Z^v = \begin{pmatrix} 0 \\ Z^A \end{pmatrix} = Z^A \frac{\partial}{\partial v^A} .$$

Hence in the case under consideration one has

$$T^v = \sum t^A \frac{\partial}{\partial v^A} ; \quad A = 0, 1, \dots, 2m \quad (4.14)$$

and by (4.9) one gets

$$i_{T^v} \psi = \omega . \quad (4.15)$$

Therefore by (4.10) one derives

$$L_{T^v} \psi = 0 \quad (4.16)$$

and one may say that T^v defines an infinitesimal automorphism of ψ .

Finally we set

$$r = f v \quad (4.17)$$

where

$$v = \frac{1}{2} \sum (v^A)^2 \quad (4.18)$$

denotes the Liouville function on M ([9]).

Operating on r by the vertical differentiation operator d_v ([7]) one gets

$$d_v r = f \sum_A v^A \omega^A = f \mu \quad (4.19)$$

and taking the exterior differential of (4.19) we obtain by (2.13) and (4.9)

$$d(d_v r) = f \sum d v^A \wedge \omega^A = f \psi . \quad (4.20)$$

Next putting $II = f \psi$ it follows by (2.13)

$$dII = 0 . \quad (2.21)$$

Therefore the exact symplectic form II can be viewed as the canonical symplectic form of the $(4m + 2)$ -dimensional manifold TM ([13]).

Finally by reference to [13] one may consider that the pair (r, II) defines a regular mechanical system $\mathcal{M}$ (in the sense of Klein [13]) having the scalar r as kinetic energy.

THEOREM. Let TM be the tangent bundle manifold having as basis the exact (L.C.C.)-manifold $M(\Omega, T, \omega)$ discussed in Section 2. Let V, γ and ν be the Liouville vector field, the Liouville form and the Liouville function of TM , respectively. One has the following properties:

- i) the complete lift Ω^c on TM of the conformal cosymplectic form Ω of M is a homogeneous of class 1, 2-form, i.e. $L_V \Omega^c = \Omega^c$, and it is $d^{-\omega}$ -exact, i.e. $d^{-\omega} \Omega^c = 0$;
- ii) γ satisfies $d^{-\omega} \gamma = \psi \Rightarrow d^{-\omega} \psi = 0$ and ψ is the canonical conformal symplectic form of TM and ψ enjoys also the property to be a Finslerian form;
- iii) the vertical lift $T^* \nu$ of ν defines an infinitesimal automorphism of ψ , i.e. $L_{T^* \nu} \psi = 0$;
- iv) $r = f\nu$ and $f\psi$ define a regular mechanical system on TM having r as kinetic energy and $f\psi$ as canonical symplectic form (where f is the energy function of M).

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