EXISTENCE OF SOLUTIONS FOR A NONLINEAR HYPERBOLIC-PARABOLIC EQUATION IN A NON-CYLINDER DOMAIN

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ABSTRACT. In this paper, we study the existence of global weak solutions for the equation

$$k_2(x)u'' + k_1(x)u' + A(t)u + |u|^{\rho}u = f \quad (\text{I})$$

in the non-cylinder domain Q in $\mathbb{R}^{n+1}$; k_1 and k_2 are bounded real functions, $A(t)$ is the symmetric operator

$$A(t) = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a_{ij}(x,t) \frac{\partial}{\partial x_i} \right)$$

where a_{ij} and f are real functions given in Q . For the proof of existence of global weak solutions we use the Faedo-Galerkin method, compactness arguments and penalization.

KEY WORDS AND PHRASES. Existence of weak solutions, Faedo-Galerkin method, compactness arguments.

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INTRODUCTION AND TERMINOLOGY.

Let $T \geq 0$ be a positive real number, O a bounded open set of $\mathbb{R}^n$ and $Q \subset O \times [0, T)$ a non-cylindrical domain in $\mathbb{R}^{n+1}$.

In the cylinder $\Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set, Bensoussan et al. [1] and Lions [7] have studied the homogenization for the following Cauchy problem:

$$\begin{aligned} k_2(x)u'' + k_1(x)u' + \Delta u &= f \text{ in } \Omega \\ u(x, 0) &= u_0(x) \text{ and } k_2(x)u'(x, 0) = k_2^{1/2}(x)u_1(x), x \in \Omega \end{aligned} \quad (\text{II})$$

Many authors have been investigating the solvability of solution for the nonlinear equations associated with problem (I) see: Larkin [4], Lima [5], Medeiros [9], Medeiros [10], Medeiros [11], Melo [12], Maciel [13], Neves [14] and Vagrov [16].

In the non-cylindrical domain Q , Lions, J.L. [8] studied the existence and uniqueness of global weak solutions for nonlinear equations associated with problem (II) with nonlinearity of type $|u|^{\rho}u$.

Let $\Omega_t = Q \cap \{t = s\}$ be a plane in $\mathbb{R}^{n+1}$. Analogously $\Omega_0 = Q \cap \{t = 0\}$ and $\Omega_T = Q \cap \{t = T\}$; $\partial Q = \Gamma$ the boundary of Q ; $\Gamma_s = \partial Q \cap \{t = s\}$ the boundary de Ω_s and $\Sigma = \cup_{0 < s < T} \Gamma_s$ lateral boundary of Q . Therefore Q is a subset of $O \times (0, T)$ whose boundary is $\Omega_0 \cap \Sigma \cap \Omega_T$.

Let's denote by $(\cdot, \cdot)$ and $|\cdot|$ the inner product and the norm in $L^2(\Omega)$ and by $((\cdot, \cdot))$ and

$\| \cdot \|$ the inner product and norm in $H_0^1(\Omega)$. We identify $L^2(\Omega_t)$ and $H_0^1(\Omega_t)$ the sub-space of the $L^2(O)$ and $H_0^1(O)$ respectively, $\forall 0 \leq t \leq T$.

We define $L^p(0, T; L^2(\Omega_t))$ to be the space of functions v in $L^p(0, T; L^2(O))$ such that $v(t)$ in $L^2(\Omega_t)$ a.e. on t , for $1 \leq p \leq \infty$. By analogy we define $L^p(0, T; H_0^1(\Omega_t))$.

In this work we study the following problem: Let $f, k_1, k_2, u_0 \in u_1$ be functions in appropriate spaces. We want to find the function $u: Q \rightarrow \mathbb{R}$ such that:

$$k_2(x)u'' + k_1(x)u' + A(t)u + \|u\|^p u = f \text{ in } Q, \text{ with } 0 < p \in \mathbb{R}, \text{ where}$$

$$u(x, 0) = u_0(0), u_t(x, 0) = u_1(x) \text{ in } \Omega_0 \text{ and}$$

$$A(t) = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a_{ij}(x, t) \frac{\partial}{\partial x_i} \right) \text{ with } a_{ij} \text{ in } Q$$

We use Faedo-Galerkin's method and compactness arguments, see Lions, J.L. [7]

1. ASSUMPTIONS AND MAIN RESULT.

If we assume the following hypothesis:

(H.1) Let Ω_t^* be the projection of the Ω_t on the hyperplane $t = 0$. We may assume $\Omega_t^* \subset \Omega_s^*$ if $t \leq s$.

(H.2) For each $t \in [0, T]$, Ω_t has the following regularity: If $u \in H_1(O)$ and $u = 0$ in $O - \Omega_t^*$ a.e., then the restriction of u to Ω_t belongs to $H_0^1(\Omega_t)$.

On the functions k_1, k_2 and a_{ij} we take:

(H.3) $k_1, k_2 \in L^\infty(\Omega_t); k_1(x) \geq \beta > 0, \beta \in \mathbb{R}; k_2(x) \geq 0$ for each $t \in [0, T]$.

(H.4) $a_{ij} = a_{ji} \in L^\infty(O \times (0, T))$ and $a'_{ij} = \frac{\partial}{\partial t} a_{ij} \in L^\infty(O \times (0, T))$.

There is $0 < \delta \in \mathbb{R}$ such that

$$\sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \geq \delta (|\xi_1|^2 + \dots + |\xi_n|^2), (x, t) \in O \times (0, T), \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

Let $a(t, u, v)$ denote the bilinear form associated to the operator $A(t)$. From (H.4) and, using Cauchy-Schwartz, we obtain:

$$a(t, u, v) \leq C \|u\| \cdot \|v\|; \forall u, v \in H_0^1(O).$$

Also by Poincaré-Friedrichs inequality and of (H.4), there exists $\alpha > 0$, real, such that:

$$a(t, u, v) \geq \alpha \|u\|^2; \forall u \in H_0^1(O).$$

Therefore, from the above inequalities, we conclude that $a(t, \cdot, \cdot)$ is continuous and coercive in $H_0^1(O) \times H_0^1(O)$.

Now lets consider the main result.

THEOREM 1. Suppose the hypothesis (H.1)-(H.4) are satisfied and that

$$f \in L^2(Q) \tag{1.1}$$

$$u_0 \in H_0^1(\Omega_0) \tag{1.2}$$

$$u_1 \in L^2(\Omega_o) \text{ are given, with } 0 < p \leq \frac{4}{n-2} \tag{1.3}$$

Then there exists a function $u: Q \rightarrow \mathbb{R}$ such that

$$u \in L^\infty(0, T; H_0^1(\Omega_t)) \tag{1.4}$$

$$u' \in L^\infty(0, T; L^2(\Omega_t)), \sqrt{k_2(x)}u' \in L^\infty(0, T; L^2(\Omega_t)) \tag{1.5}$$

$$k_2(x)u'' \in L^{p'}(0, T; H^{-1}(\Omega_t)) \text{ with } \frac{1}{p} + \frac{1}{p'} = 1, p = p + 2 \text{ and } u \tag{1.6}$$

is a solution (1) in the weak sense in Q , i.e.,

$$\frac{d}{dt} (k_2(x)u_t(t), v) + (k_1(x)u_t(t), v) + a(t, u(t), v) + (|u(t)|^\rho u(t), v) = (f(t), v), \quad (1.7)$$

in $D'(0, T)$, $\forall v \in H_0^1(\Omega_t)$.

$$u(x, 0) = u_0(x); \quad k_2(x)u_t(x, 0) = \sqrt{k_2(x)}u_1 \text{ in } \Omega_0 \quad (1.8)$$

PROOF. The idea is to transform, the non-cylinder problem in the cylinder problem, through the penalization function, $M \in L^\infty(O \times (0, T))$, that was introduced by J.L. Lions [8], given by:

$$M(x, t) = \begin{cases} 0, & \text{in } Q \\ 1, & \text{in } O \times (0, T) \setminus Q. \end{cases}$$

For each $\epsilon > 0$, we will find U^ϵ in the cylinder $O \times (0, T)$, solution of the perturbed problem (P_ϵ) below

$$\tilde{k}_{2\epsilon}(x)U_{tt}^\epsilon + \tilde{k}_1(x)U_t^\epsilon + A(t)U^\epsilon + \frac{1}{\epsilon}MU_t^\epsilon + \|U^\epsilon\|^\rho U^\epsilon = \tilde{f} \quad (1.9)$$

$$U^\epsilon(0) = \tilde{u}_0 \quad (1.10)$$

$$\tilde{k}_{2\epsilon}U_t^\epsilon(0) = \sqrt{\tilde{k}_{2\epsilon}(x)}\tilde{u}_1 \quad (1.11)$$

$$U^\epsilon = 0 \text{ in the } \partial(O \times (0, T)) = \tilde{\Sigma} \quad (1.12)$$

where $\tilde{k}_{2\epsilon}(x) = \tilde{k}_2(x) + \epsilon$; $U_t = \frac{\partial}{\partial t} U$; $U_{tt} = \frac{\partial^2}{\partial t^2} U$; $\tilde{u}_0 = \begin{cases} u_0 & \text{in } \Omega_0 \\ 0 & \text{in } O \setminus \Omega_0 \end{cases}$

Therefore, $\tilde{u}_0 \in H_0^1(O)$. Analogously $\tilde{u}_1 \in L^2(O)$;

$$\tilde{f} = \begin{cases} f, & \text{in } Q \\ 0, & \text{in } O \times (0, T) \setminus Q; \end{cases}$$

Therefore $\tilde{f} \in L^2(O \times (0, T))$;

$$\tilde{k}_1(x) = \begin{cases} k_1(x) & \text{in } Q \\ \beta & \text{in } O \times (0, T) \setminus Q \end{cases} \quad \text{and} \quad \tilde{k}_2(x) = \begin{cases} k_2(x) & \text{in } Q \\ 0 & \text{in } O \times (0, T) \setminus Q \end{cases}$$

So $\tilde{k}_1$ and $\tilde{k}_2 \in L^\infty(O \times (0, T))$.

The proof of Theorem 1 will be a consequence of the following Theorem:

THEOREM 2. For each $\epsilon > 0$, there exists one function $U_\epsilon: O \times (0, T) \rightarrow \mathbb{R}$, solution of the problem (P_ϵ), such that:

$$U^\epsilon \in L^\infty(0, T; H_0^1(O)) \quad (1.13)$$

$$U^\epsilon \in L^\infty(0, T; L^2(O)), \sqrt{\tilde{k}_{2\epsilon}(x)}U_t^\epsilon \in L^\infty(0, T; L^2(O)) \quad (1.14)$$

$$\tilde{k}_{2\epsilon}(x)U_{tt}^\epsilon \in L^{p'}(0, T; H^{-1}(O)) \quad (1.15)$$

with $\frac{1}{p} + \frac{1}{p'} = 1$ and $p = \rho + 2$

$$\tilde{k}_{2\epsilon}(x)U_{tt}^\epsilon + \tilde{k}_1(x)U_t^\epsilon + A(t)U^\epsilon + \frac{1}{\epsilon}MU_t^\epsilon + |U^\epsilon|^\rho U^\epsilon = \tilde{f} \quad (1.16)$$

in the weak sense in $O \times (0, T)$.

$$U^\epsilon(x, 0) = \tilde{u}_0(x) \quad (1.17)$$

$$\tilde{k}_{2\epsilon}(x)U_t^\epsilon(x, 0) = \sqrt{\tilde{k}_{2\epsilon}(x)}\tilde{u}_1(x) \quad (1.18)$$

REMARK 1. The condition $U^\epsilon = 0$ in $\dot{\Sigma}$ is a consequence of the fact that U^ϵ in $L^2(0, T; H_0^1(O))$.

REMARK 2. For the proof of Theorem 1 it is sufficient to prove that the solution U^ϵ in Theorem 2 converges for U in the weak sense when $\epsilon \rightarrow 0$ and that the restriction of U to Q satisfies all the assertions of Theorem 1.

In this part, we use a result due to W.A. Strauss see [15].

PROOF OF THEOREM 2.

(i) **Approximate Problem.** It will be done by the Faedo-Galerkin method. Let $\{w_\nu\}_{\nu=1}^\infty \subset H^1(O)$ be a basis of $H^1(O)$ and V_m the subspace spanned by the m first vectors $w_1, w_2, \dots, w_m$. Let U_m^ϵ be the function

$$U_m^\epsilon(x, t) = \sum_{j=1}^m g_{jm\epsilon}(t) w_j(x)$$

defined by the system

$$\begin{aligned} (\dot{k}_{2\epsilon}(x) \frac{\partial^2}{\partial t^2} U_m^\epsilon(t), w_j) + (\dot{k}_1(x) \frac{\partial}{\partial t} U_m^\epsilon(t), w_j) + a(t, U_m^\epsilon(t), w_j) \\ + \frac{1}{\epsilon} M \left(\frac{\partial}{\partial t} U_m^\epsilon(t), w_j \right) + (|U_m^\epsilon(t)|^p, w_j) = (f(t), w_j), \quad \forall j = 1, \dots, m \end{aligned} \quad (1.19)$$

$$U_m^\epsilon(0) = U_{0m} = \sum_{j=1}^m \alpha_{jm} w_j \rightarrow \tilde{u}_0 \text{ strong in } H^1(O) \quad (1.20)$$

$$\frac{\partial}{\partial t} U_m^\epsilon(0) = U_{1m} = \sum_{j=1}^m \beta_{jm} w_j \rightarrow \frac{\tilde{u}_1}{\sqrt{k_{2\epsilon}}} \text{ strong in } L^2(O) \quad (1.21)$$

The system (1.19)-(1.21) satisfies the condition of Caracetheodory's theorem see [2]. Therefore it has a solution U_m^ϵ defined in $[0, t_{\epsilon m})$, where $0 < t_{\epsilon m} \leq T$. The a priori estimates to be obtained in the following step, show, in particular, that $t_{\epsilon m} = T$.

(ii) **A Priori Estimates.** By multiplying both sides of (1.19) by $2g'_{jm\epsilon}(t)$, and adding from $j = 1$ to $j = m$ we obtain:

$$\begin{aligned} \frac{d}{dt} \left| \sqrt{k_{2\epsilon}(x)} U_m'(t) \right|^2 + 2 \left| \sqrt{k_1(x)} U_m(t) \right|^2 + 2a(t, U_m(t), U_m'(t)) + \frac{2}{\epsilon} \int_O M(U_m')^2 dx \\ + 2 \int_O \|U_m(s)\|^p U_m(s) U_m'(s) dx = 2(\tilde{f}(t), U_m'(t)), \end{aligned} \quad (1.22)$$

where we wrote U_m instead of U_m^ϵ and denoted by $U_m' = \frac{\partial}{\partial t} U_m$.

REMARK 3. We have that

$$\frac{d}{dt} a(t, U_m(t), U_m(t)) = a'(t, U_m(t), U_m(t)) + 2a(t, U_m(t), U_m'(t));$$

where

$$a'(t, U_m(t), U_m(t)) = a'(t, U_m(t)) = \sum_{i,j=1}^n \int_O \frac{\partial}{\partial t} a_{ij}(x, t) \frac{\partial}{\partial x_i} U_m(t) \frac{\partial}{\partial x_j} U_m(t) dx.$$

Therefore,

$$2a(t, U_m(t), U_m'(t)) = \frac{d}{dt} a(t, U_m(t)) - a'(t, U_m(t)).$$

REMARK 4. We have that $\frac{1}{p} \frac{d}{dt} \int_O |U_m(s)|^p dx = \int_O |U_m(s)|^{p-1} \cdot \frac{U_m(s)}{|U_m(s)|} \cdot U_m'(s) dx = \int_O |U_m(s)|^p U_m'(s) dx$.

Therefore, in the remarks (3 and 4) below, we have, integrating (1.22) from 0 to t , $0 < t \leq t_m$, that:

$$\left| \sqrt{k_{2\epsilon}(x)} U_m'(t) \right|^2 + 2 \int_0^t \left| \sqrt{k_1(x)} U_m'(s) \right|^2 ds + a(t, U_m(t)) + \frac{2}{p} \int_O |U_m(s)|^p dx$$

$$+ \frac{2}{\epsilon} \int_0^t \int_O M(U'_m(s))^2 dx ds = \left| \sqrt{\tilde{k}_{2\epsilon}(x)} U_{1m} \right|^2 + a(0, U_{0m}) + \quad (1.23)$$

$$\frac{2}{p} \int_O |U_{0m}|^p dx + \int_0^t a'(s, U_m(s)) ds + 2 \int_0^t (\dot{f}(s), U'_m(s)) ds$$

REMARK 5. From (20), (21) and the Sobolev Immersion, $H^1(O) \hookrightarrow L^p(O)$, $\forall \frac{1}{p} = \frac{1}{2} - \frac{1}{n}$, we obtain:

$$\|U_{0m}\|_{L^p(O)} \leq C.$$

$$\left| \sqrt{\tilde{k}_{2\epsilon}(x)} U_{1m} \right|^2 \leq C; \quad |a(0, U_{0m})| \leq C.$$

Here, the letter C denotes different constants.

REMARK 6. By using (H.4), we obtain:

$$\int_0^t |a'(s, U_m(s))| ds \leq C \int_0^t \|U_m(s)\|^2 ds;$$

Therefore, from the remarks (5 and 6) below, we can write (1.23) like

$$\begin{aligned} & \left| \sqrt{\tilde{k}_{2\epsilon}(x)} U'_m(t) \right|^2 + 2 \int_0^t \left| \sqrt{\tilde{k}_1(x)} U'_m(s) \right|^2 ds + a(t, U_m(t)) + \frac{2}{p} \int_O |U_m(s)|^p ds \\ & + \frac{2}{\epsilon} \int_0^t \int_O M(U'_m(s))^2 dx ds \leq C + C \int_0^t \|U_m(s)\|^2 ds + \lambda \int_0^t |U'_m(s)|^2 ds \end{aligned} \quad (1.24)$$

From (1.24), if we choose $\lambda = \beta > 0$ (the $\beta > 0$ of H.3) we obtain:

$$\int_0^t |U'_m(s)|^2 ds \leq C + C \int_0^t \|U_m(s)\|^2 ds, \quad (1.25)$$

and

$$a(t, U_m(t)) \leq C + C \int_0^t \|U_m(s)\|^2 ds + \beta \int_0^t |U'_m(s)|^2 ds \quad (1.26)$$

Being $a(t, u, v)$ coercive, we obtain from (1.25) and (1.26), that:

$$\|U_m(t)\|^2 \leq C + C \int_0^t \|u_m(s)\|^2 ds, \quad \forall t \in [0, t_{\epsilon m}]. \quad (1.27)$$

Gronwall's inequality implies that

$$\|U_m^\epsilon\| \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \epsilon > 0, \quad \forall t \in [0, t_{\epsilon m}]. \quad (1.28)$$

Returning to (1.25) we obtain:

$$\int_0^t \left| \frac{\partial}{\partial s} U_m^\epsilon(s) \right|^2 ds \leq C, \quad (1.29)$$

$\forall m \in \mathbb{N}, \quad \forall \epsilon > 0, \quad \forall t \in [0, t_{\epsilon m}].$

The priori estimative (1.24) shows that $t_{\epsilon m} = T$. Therefore,

$$\begin{aligned} & \left| \sqrt{\tilde{k}_{2\epsilon}(x)} \frac{\partial}{\partial s} U_m^\epsilon(t) \right|^2 + 2 \int_0^t \left| \sqrt{\tilde{k}_1(x)} \frac{\partial}{\partial s} U_m^\epsilon(s) \right|^2 ds + a(t, U_m^\epsilon(t)) \\ & + \frac{2}{p} \int_O |U_m^\epsilon(s)|^p dx + \frac{2}{\epsilon} \int_0^t \int_O M \left(\frac{\partial}{\partial s} U_m^\epsilon(s) \right)^2 dx ds \leq C \end{aligned} \quad (1.30)$$

$\forall m \in \mathbb{N}, \quad \forall \epsilon > 0$ and $\forall t \in [0, T]$.

We obtain from (1.28), (1.29) and (1.30) the estimates,

$$\|U_m^\epsilon\|_{L^\infty(0, T; H^1_0(O))} \leq C, \quad \forall m \in \mathbb{N}, \quad \epsilon > 0. \quad (1.31)$$

$$\left\| \frac{\partial U_m^\epsilon}{\partial t} \right\|_{L^2(0,T;L^2(O))} \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \epsilon > 0 \quad (1.32)$$

$$\left\| \sqrt{k_{2\epsilon}} \frac{\partial U_m^\epsilon}{\partial t} \right\|_{L^\infty(0,T;L^2(O))} \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \epsilon > 0 \quad (1.33)$$

$$\left\| \frac{1}{\sqrt{\epsilon}} M \frac{\partial U_m^\epsilon}{\partial t} \right\|_{L^\infty(0,T;L^2(O))} \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \epsilon > 0: \quad (1.34)$$

where C is a constant independent of $m \in \mathbb{N}$ and $\epsilon > 0$.

By the estimates (1.31)-(1.34), there exist a subsequence of (U_m^ϵ) , still denoted by (U_m^ϵ) , and a function U^ϵ such that

$$U_m^\epsilon \rightarrow U^\epsilon \text{ weak-star in } L^\infty(0,T;H_0^1(O)), \quad (1.35)$$

$$\frac{\partial}{\partial t} U_m^\epsilon \rightarrow \frac{\partial}{\partial t} U^\epsilon \text{ weak in } L^2(0,T;L^2(O)), \quad (1.36)$$

$$\frac{1}{\sqrt{\epsilon}} M \frac{\partial}{\partial t} U_m^\epsilon \rightarrow \frac{1}{\sqrt{\epsilon}} M \frac{\partial}{\partial t} U^\epsilon \text{ weak-star in } L^\infty(0,T;L^2(O)). \quad (1.37)$$

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By (1.30) and noting that $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain

$$\| |U_m^\epsilon|^\rho U_m^\epsilon \|_{L^{p'}}^{p'} = \int_O |U_m^\epsilon|^{(\rho+1)p'} dx = \int_O |U_m^\epsilon|^{(p-1)p'} dx = \int_O |U_m^\epsilon|^p dx \leq C,$$

which implies:

$$\| |U_m^\epsilon|^\rho U_m^\epsilon \|_{L^\infty(0,T;L^{p'}(O))} \leq C, \quad \forall m \in \mathbb{N}, \quad \forall \epsilon > 0. \quad (1.38)$$

From (1.31), (1.32) and the Aubin-Lions Theorem (see [7]) we obtain:

$$|U_m^\epsilon|^\rho U_m^\epsilon \rightarrow |U^\epsilon|^\rho U^\epsilon \text{ a.e. in } O \times (0,T), \quad (1.39)$$

and

$$|U_m^\epsilon|^\rho U_m^\epsilon \rightarrow W \text{ weak-star in } L^\infty(0,T;L^{p'}(O)) \quad (1.40)$$

The difficulty is to prove that $W = |U^\epsilon|^\rho U^\epsilon$. This is a consequence of the following result due to W.A. Strauss (see [15]).

LEMMA 1. Let Ω be a bounded open set of $\mathbb{R}^n$. Lets g_m and $g \in L^p(\Omega)$, $1 < p < \infty$ satisfy the following conditions:

- (i) $g_m \rightarrow g$ a.e. in Ω
- (ii) $\|g_m\|_{L^p(\Omega)} \leq C, \quad \forall m \in \mathbb{N}$.

Then

- (iii) $g_m \rightarrow g$ strongly in $L^q(\Omega)$, $1 \leq q < p$
- (iv) $g_m \rightarrow g$ weakly in $L^p(\Omega)$.

Lemma 1 with $q = \frac{p+2}{p-1} = p'$, $\Omega = O \times (0,T)$ and $g_m = |U_m|^\rho U_m$, we obtain from (1.38) and (1.39) that

$$|U_m^\epsilon|^\rho U_m^\epsilon \rightarrow |U^\epsilon|^\rho U^\epsilon \text{ weak-start in } L^\infty(0,T;L^{p'}(O)) \quad (1.41)$$

and consequently weak in $L^{p'}(0,T;L^{p'}(O))$.

By multiplying both sides of (1.19) by $\theta \in C_0^\infty(0,T)$, integrating from $t = 0$ to $t = T$, passing to the limit and using the convergences (1.35)-(1.37), (1.41) and noting that $\{w_\nu\}_{\nu=1}^\infty$ is a basis of $H_0^1(O)$, we obtain:

$$\begin{aligned}
& \int_0^T (\dot{k}_{2\epsilon}(x) \frac{\partial^2}{\partial t^2} U^\epsilon(t), v\theta) dt + \int_0^T (\dot{k}_1(x) \frac{\partial}{\partial t} U^\epsilon(t), v\theta) dt + \int_0^T a(t, U^\epsilon(t), v\theta) dt + \\
& \int_0^T \left(\frac{1}{\epsilon} M \frac{\partial}{\partial t} U^\epsilon(t), v\theta \right) dt + \int_0^T (|U^\epsilon(t)|^\rho U^\epsilon(t), v\theta) dt = \int_0^T (\dot{f}(t), v\theta) dt,
\end{aligned} \tag{1.42}$$

$\forall v \in H_0^1(O), \forall \theta \in C_0^\infty(0, T)$.

Then, from (1.35)-(1.37) and from (1.42), we obtain U^ϵ satisfying (1.9)-(1.10) and (1.12). Noting that

$$L^2(0, T; L^2(O)) \hookrightarrow L^2(0, T; H^{-1}(O)),$$

we obtain

$$-\frac{1}{\epsilon} M U_t^\epsilon - \dot{k}_1(x) U_t^\epsilon \in L^2(0, T; H^{-1}(O)).$$

The fact that $a_{i,j}(x, t) \frac{\partial}{\partial x_i} U^\epsilon(t) \in L^2(O)$ implies that

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a_{i,j}(x, t) \frac{\partial}{\partial x_i} U^\epsilon \right) \in L^2(0, T; H^{-1}(O)),$$

(see [3]). Also from (1.16), (1.41) and $\tilde{f} \in L^2(0, T; L^2(O))$ we obtain

$$\dot{k}_{2\epsilon}(x) \frac{\partial^2}{\partial t^2} U^\epsilon \in L^2(0, T; H^{-1}(O)),$$

which proves (1.15).

The estimates (1.31)-(1.34) and (1.38) are independent from $\epsilon > 0$, we obtain the same convergences (1.35)-(1.37) and (1.41) by changing U_m^ϵ by U^ϵ and U^ϵ by W . Therefore, we have

$$U^\epsilon \rightharpoonup W \text{ weak-star in } L^\infty(0, T; H_0^1(O)) \tag{1.43}$$

$$U_t^\epsilon \rightharpoonup W_t \text{ weak in } L^2(0, T; L^2(O)) \tag{1.44}$$

$$\sqrt{\dot{k}_{2\epsilon}(x)} U_t^\epsilon \rightarrow \sqrt{\dot{k}_2(x)} W_t, \text{ weak-star in } L^\infty(0, T; L^2(O)). \tag{1.45}$$

Note that $\sqrt{\dot{k}_{2\epsilon}(x)} = \sqrt{\dot{k}_2(x)} + \epsilon \rightarrow \sqrt{\dot{k}_2(x)}$ strong in $L^2(0, T; L^2(\Omega))$.

$$|U^\epsilon|^\rho U^\epsilon \rightharpoonup |W|^\rho W \text{ weak-star in } L^\infty(0, T; L^\rho(O)) \tag{1.46}$$

Also, we obtain the essential estimates:

$$\int_{O \times (0, T)} M(U_t^\epsilon) dx dt \leq C\epsilon. \tag{1.47}$$

From (1.44) we have: $M(U_t^\epsilon)^2 \rightarrow M(W_t)^2$ weak in $L^2(0, T; L^2(O))$.

Therefore, from (1.47) we obtain

$$\int_{O \times (0, T)} M(W_t)^2 dx dt = 0.$$

From this and the definition of M , we deduce: $W_t = 0$ a.e. in $O \times (0, T) \supset Q$. Consequently $W(x, t)$ is constant in the variable t in $O \times (0, T) \supset Q$. Being $W(x, 0) = \tilde{u}_0(x)$ in O , we conclude that $W(x, 0) = 0$ in $O \setminus \Omega_0$. From this and from (H-1), we get:

$$W(x, t) = 0 \text{ a.e. in } O \times (0, T) \supset Q. \tag{1.48}$$

We conclude from (1.43) and (1.44) that $W(t) \in H^1(O)$. Let u be the restriction of W to Q .

Then from (1.48) and (H-2), we obtain that $u \in L^\infty(0, T; H_0^1(\Omega_t))$; which proves (1.4) in Theorem

1. Moreover, from (1.44) and (1.45), we conclude that u satisfies (1.5).

Let $\widehat{U}$ be the restriction of U to Q . Then, restricting the equation of Theorem 2 to the domain Q , we obtain:

$$\begin{aligned} & (k_{2\epsilon}(x)\widehat{U}_t^\epsilon(t), v) + (k_1(x)\widehat{U}_t^\epsilon(t), v) + a(t, \widehat{U}^\epsilon(t), v) + \frac{1}{\epsilon}(M\widehat{U}_t^\epsilon(t), v) + \\ & (|\widehat{U}^\epsilon(t)|^\rho \widehat{U}^\epsilon(t), v) = (\tilde{f}(t), v), \end{aligned} \quad (1.49)$$

$\forall v \in H_0^1(O)$, in the sense of the $D'(0, T)$.

By taking the limit when $\epsilon \rightarrow 0$ in (1.49), and using the convergences (1.43)-(1.46) we get:

$$\frac{d}{dt} (k_2(x)u_t(t), v) + (k_1(x)u_t(t), v) + a(t, u(t), v) + (|u(t)|^\rho u(t), v) = (f(t), v),$$

in $D'(0, T)$, $\forall v \in H_0^1(\Omega_t)$, which proves (1.7).

The proof of (1.6) is analogous to (1.15) of the cylinder problem.

(iii) The Initial Conditions.

Let $\sigma \in C^1([0, T]; \mathbb{R})$ be such that $\sigma(0) = 1$ and $\sigma(T) = 0$. We have

$$\int_0^T \left(\frac{\partial}{\partial t} U_m^\epsilon(t), v \right) \sigma(t) dt = - (U_m^\epsilon(0), v) - \int_0^T (U_m^\epsilon(t), v) \sigma'(t) dt, \quad \forall v \in L^2(O).$$

By passing to the limit in the above equality and using the convergences (1.20), (1.35) and (1.36) we obtain:

$$\int_0^T \left(\frac{\partial}{\partial t} U^\epsilon(t), v \right) \sigma(t) dt = - (\tilde{u}_0, v) - \int_0^T (U^\epsilon(t), v) \sigma'(t) dt, \quad \forall v \in L^2(O).$$

Integrating by parts the last integral above, we conclude that

$$(U^\epsilon(0), v) = (\tilde{u}_0, v), \quad \forall v \in L^2(O).$$

From this it follows (1.17). The initial condition $u(x, 0) = u_0(x)$ of Theorem 1 is done analogously.

Finally, we will verify condition (1.18). Initially we verify that $[(k_2(x) + \epsilon)U_\epsilon](0)$ does make sense.

Let U^ϵ be a solution of the perturbed problem. Then

$$\begin{aligned} & - \int_0^T \left\langle \tilde{k}_{2\epsilon}(x)U_t^\epsilon(t), \theta'(t)v \right\rangle dt + \int_0^T \left\langle \tilde{k}_1(x)U_t^\epsilon(t), \theta(t)v \right\rangle dt + \int_0^T \left\langle A(t)U^\epsilon(t), \theta(t)v \right\rangle dt + \\ & \int_0^T \left\langle \frac{1}{\epsilon} M U_t^\epsilon(t), \theta(t)v \right\rangle dt + \int_0^T \left\langle |U^\epsilon(t)|^\rho U^\epsilon(t), \theta(t)v \right\rangle dt = \int_0^T \left\langle \tilde{f}(t), \theta(t)v \right\rangle dt \end{aligned}$$

$\forall v \in H_0^1(O)$ and $\forall \theta \in C_0^\infty(0, T)$; where $\langle \cdot, \cdot \rangle$ is the duality between $H_0^1(O)$ and $H^{-1}(O)$. So

$$\begin{aligned} & \left\langle - \int_0^T \tilde{k}_{2\epsilon}(x)U_t^\epsilon(t)\theta'(t)dt + \int_0^T \tilde{k}_1(x)U_t^\epsilon(t)\theta(t)dt + \right. \\ & \left. \int_0^T A(t)U^\epsilon(t)\theta(t)dt + \int_0^T \frac{1}{\epsilon} M U_t^\epsilon(t)\theta(t)dt + \right. \\ & \left. \int_0^T |U^\epsilon(t)|^\rho U^\epsilon(t)\theta(t)dt, v \right\rangle = \left\langle \int_0^T \tilde{f}(t)\theta(t)dt, v \right\rangle \end{aligned}$$

$\forall v \in H_0^1(O)$ and $\forall \theta \in C_0^\infty(0, T)$.

Therefore, we have

$$\begin{aligned} & < -\dot{k}_{2\epsilon}(x)U'_i(t), \theta'(t) > + < \dot{k}_1(x)U'_i(t), \theta(t) > + < A(t)U''(t), \theta(t) > + \\ & < \frac{1}{\epsilon} MU'_i(t), \theta(t) > + < |U^\epsilon(t)|^\rho U^\epsilon(t), \theta(t) > = < \tilde{f}(t), \theta(t) >. \end{aligned}$$

$\forall \theta \in C_0^\infty(0, T)$; where, here $< \cdot, \cdot >$ denotes the vectorial distribution of $(0, T)$ in $H^{-1}(O)$ evaluated in scalar test application of $(0, T)$. Being $\dot{k}_{2\epsilon} \in L^\infty(O \times (0, T))$ and $U'_i \in L^2(0, T; L^2(O))$, we have $-\dot{k}_{2\epsilon}U'_i \in L^2(0, T; L^2(O))$.

So $-\dot{k}_{2\epsilon}U'_i$ defines a vectorial distribution of $(0, T)$ in $L^2(O)$, whose derivative is:

$$< -\dot{k}_{2\epsilon}U'_i, \theta' > = < (\dot{k}_{2\epsilon}U'_i)_t, \theta >, \quad \forall \theta \in C_0^\infty(0, T).$$

Therefore,

$$\begin{aligned} & < (\dot{k}_{2\epsilon}U'_i)_t, \theta > + < \dot{k}_1U'_i, \theta > + < A(t)U'', \theta > + \\ & < \frac{1}{\epsilon} MU'_i, \theta > + < |U^\epsilon|^\rho U^\epsilon, \theta > = < \tilde{f}, \theta >, \quad \forall \theta \in C_0^\infty(0, T). \end{aligned}$$

Or,

$$(\dot{k}_{2\epsilon}U'_i)_t + \dot{k}_1U'_i + A(t)U'' + \frac{1}{\epsilon} MU'_i + |U^\epsilon|^\rho U^\epsilon = \tilde{f},$$

in $L^2(0, T; H^{-1}(O))$. As $\tilde{f}, \dot{k}_1U'_i, \frac{1}{\epsilon} MU'_i, |U^\epsilon|^\rho U^\epsilon \in L^2(0, T; L^2(O))$ and $A(t)U'' \in L^2(0, T; H^{-1}(O))$, we obtain, from the last equality above that: $(\dot{k}_{2\epsilon}U'_i)_t \in L^2(0, T; H^{-1}(O)) \rightarrow L^p(0, T; H^{-1}(O))$, which proves (1.15). It is easy to see that $\dot{k}_{2\epsilon}U'_i \in C^0([0, T]; H^{-1}(O))$. Therefore, $[\dot{k}_{2\epsilon}U'_i](0)$ makes sense. Let now $\theta \in C^1([0, t]; \mathbb{R})$ be such that $\theta(0) = 1$ and $\theta(T) = 0$. Then,

$$\begin{aligned} \int_0^T (\dot{k}_{2\epsilon} \frac{\partial^2}{\partial t^2} U_m^\epsilon(t), v) \theta(t) dt &= - \left(\dot{k}_{2\epsilon} \frac{\partial}{\partial t} U_m^\epsilon(0), v \right) \\ &\quad - \int_0^T \left(\dot{k}_{2\epsilon} \frac{\partial}{\partial t} U_m^\epsilon(t), v \right) \theta'(t) dt, \quad \forall v \in V_m. \end{aligned}$$

From this and taking $v = w$, in the approximate equation, we obtain:

$$\begin{aligned} & - \left(\dot{k}_{2\epsilon} \frac{\partial}{\partial t} U_m^\epsilon(0), v \right) - \int_0^T \left(\dot{k}_{2\epsilon} \frac{\partial}{\partial t} U_m^\epsilon(t), v \right) \theta'(t) dt + \int_0^T \left(\dot{k}_1 \frac{\partial}{\partial t} U_m^\epsilon(t), v \right) \theta(t) dt + \\ & \int_0^T a(t, U_m^\epsilon(t), v) \theta'(t) dt + \int_0^T \left(\frac{1}{\epsilon} M \frac{\partial}{\partial t} U_m^\epsilon(t), v \right) \theta(t) dt + \int_0^T (|U_m^\epsilon(t)|^\rho U_m^\epsilon(t), v) \theta(t) dt = \\ & \int_0^T (\tilde{f}(t), v) \theta(t) dt, \quad \forall v \in V_m. \end{aligned}$$

By passing to the limit in the above equality and using the convergences (1.21), (1.35)-(1.37) and (1.41) we obtain:

$$\begin{aligned} & - \left(\sqrt{\dot{k}_{2\epsilon}} \tilde{u}_1, v \right) - \int_0^T (\dot{k}_{2\epsilon} U_i^\epsilon(t), v) \theta'(t) dt + \int_0^T (\dot{k}_1(x) U_i^\epsilon(t), v) \theta(t) dt + \\ & \int_0^T a(t, U^\epsilon(t), v) \theta(t) dt + \int_0^T \left(\frac{1}{\epsilon} M U_i^\epsilon(t), v \right) \theta(t) dt + \\ & \int_0^T (|U^\epsilon(t)|^\rho U^\epsilon(t), v) \theta(t) dt = \int_0^T (\tilde{f}(t), v) \theta(t) dt, \end{aligned}$$

As $-\int_0^T (\dot{k}_{2\epsilon} U'_t(t), v) \theta'(t) dt = \langle (\dot{k}_{2\epsilon} U'_t(t))_t, v \rangle \theta(t) \forall v \in V_m$ and $\theta \in C^1([0, T]; \mathbb{R})$ such that $\theta(0) = 1$ and $\theta(T) = 0$, we have, using the fact that U^* is solution of the perturbed equation, that:

$$- \langle \sqrt{\dot{k}_{2\epsilon}}(x) \dot{u}_1, v \rangle + \langle \dot{k}_{2\epsilon}(x) U'_t(0), v \rangle = 0, \forall v \in V_m.$$

Or,

$$\langle \dot{k}_{2\epsilon}(x) U'_t(0) - \sqrt{\dot{k}_{2\epsilon}}(x) \dot{u}_1, v \rangle = 0,$$

$\forall v \in H_0^1(\Omega)$. This proves (1.18) and, therefore, the proof of Theorem 2 is complete.

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