THE FIXED POINT INDEX FOR ACCRETIVE MAPPING* WITH K—SET CONTRACTION PERTURBATION IN CONES

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ABSTRACT: Let P be a cone in Banach space E. A, K are two mappings in P, A is accretive, K is k—set contraction, then a fixed point index is defined for mapping —A+K, some fixed point theorems are also deduced.

KEY WORDS AND PHRASES: accretive mapping, k—set contraction, cone, fixed point index.

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1. INTRODUCTION

The fixed point index is an important tool in solving positive solutions of nonlinear equations in ordered Banach space. So what nonlinear mapping could be defined a index theory becomes a very interesting problem, many authors have studied this problem, see [3], [8], [10], [12], [13]. In this paper. E is a Banach space, P⊂E is a closed cone, i.e P is closed convex, and

\[ \lambda P \subset P, \forall \lambda \geq 0, P \cap (-P) = \{0\}; \]

\[ \Omega \subset E \] is a nonempty open bounded subset. Let \( A: D(A) \subset P \rightarrow 2^P \) be a multivalued accretive mapping, i.e

\[ \| x - y \| \leq \| x - y + \lambda(a_1 - a_2) \|, x, y \in D(A), a_1, a_2 \in A_x, a_1 \neq A_x; \]

\[ K =zp \subset P \] is a strict k—set contraction, i.e \( 0 < k < 1 \); If

\[ (I + A)(D(A)) = P, \text{ and } x \notin -Ax + Kx, \forall x \in \partial \Omega \cap D(A), \]

then a fixed point index is defined for \(-A+K\), when K is compact, such type mapping were studied by [4], [5]. [14], [15].

2. MAIN RESULTS

Let E be a Banach space. P⊂E is a closed cone, *≤* is the order induced by P in E, i.e \( x \leq y \) if and only if \( y - x \in P \).

PROPOSITION 1. \( A: D(A) \subset P \rightarrow 2^P \) is a continuous accretive mapping, for each \( x \in P \), there exists \( \beta(x) > 0 \), such that \( Ax \leq \beta(x) \cdot x \), then (\( I + A \))P = P, \( \forall \lambda > 0 \).

PROOF. For each \( x \in P \), consider the following differential equation

\[ \begin{align*}
        \dot{x}(t) &= -(I + A)x(t) + \epsilon, t \in (0, +\infty) \\
        x(0) &= u \in P
    \end{align*} \] (\( 2 \cdot 1 \))

For each \( x \in P \), since \( Ax \leq \beta(x) \cdot x \), so there exists \( W(x) \in P \), such that \( \beta(x) \cdot x = Ax + W(x) \).

So we have \( x + \epsilon(-\lambda x - Ax + z) = (1 - \epsilon\lambda - \epsilon\beta(x))x + \epsilon W(x) + \epsilon z \).

For sufficient small \( \epsilon > 0 \), such that \( 1 - \epsilon\lambda - \epsilon\beta(x) > 0 \), then \( (1 - \epsilon\lambda - \epsilon\beta(x))x + \epsilon W(x) + \epsilon z \in P \).

Hence

\[ \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \rho(x + \epsilon(-\lambda x - Ax + z), P) = 0, \forall x \in P; \]
by (6), we know (El) has only one solution. Let \( x(t,u) \) be the unique solution of (El) with \( x(0) = u \).

Now, define a mapping \( B_1 : P \to P \) as following:

\[
B_1u = x(T,u), u \in P, \quad T > 0 \text{ a constant;}
\]

For \( u, v \in P \), let \( \mathcal{D}(t) = \| x(t,u) - x(t,v) \| \), then

\[
\mathcal{D}(t) D(t) \leq \| x'(t,u) - x'(t,v) \| = \| x(t,u) - x(t,v) \|
\]

where \( D(t) = \lim_{h \to 0^+} \mathcal{D}(t) D(t-h) / h \); see (63P,36)

\[
D(t) D(t) \leq -\lambda x(t,u) - Ax(t,u) + \lambda x(t,v) + Ax(t,v) \cdot x(t,u) - x(t,v)
\]

\( \lambda \) is accretive, so

\[
D(t) D(t) \leq \| -Ax(t,u) - Ax(t,v) \| \leq 0
\]

Therefore

\[
\mathcal{D}(t) D(t) \leq \lambda x(t,u) - x(t,v)
\]

So we have \( \| B_1u - B_1v \| \leq \| u - v \| \)

Hence, \( B_1 \) has a unique fixed point \( u_0 \). This implies \( x'(t,u_0) = 0 \), \( t > 0 \).

So \( \mathcal{O} = -Au_0 - A_0 + \mathcal{O} \in \mathcal{D}(A) \).

This complete the proof.

In the following, we assume \( A : D(A) \subset P \to \mathbb{R}^n \) is a multivalued accretive mapping, \( (A+I) D(A) = P \), it's well known \( (A+I) \) is nonexpansive (see(4)).

Let \( \varnothing \) be a open bounded subset of \( E, K, \mathcal{D} \in P \) is a strict \( k \)-set contraction, i.e \( k \in (0,1) \); Suppose \( D(A) \cap \varnothing \neq \varnothing \), and \( x \in A + K, x \in \mathcal{D} \cap D(A) \), then

\[
x \notin (A+K) - K, x \notin \mathcal{D} \cap D(A)
\]

\( (A+K) \) is also a strict \( k \)-set contraction, so the fixed point index \( i((A+K), \varnothing \cap P) \) is well defined, see [1], (8). Now, we define

\[
i((A+K), \mathcal{D}(A)) = i((I+A) - K, \varnothing \cap P)
\]

THEOREM 1: (a) If \( \mathcal{O} = B(0,r), Kx = x, x \in B(0,r) \cap P \), then

\[
i((A+K), B(0,r) \cap D(A)) = 1
\]

(b) Suppose \( \mathcal{O} = \mathcal{O}, \mathcal{O} = \mathcal{O}, \mathcal{O} = \mathcal{O}, \mathcal{O} = \mathcal{O} \), then

\[
i((A+K), \mathcal{D}(A)) = i((A+K), \mathcal{D}(A)) + i((A+K), \mathcal{O} \cap D(A))
\]

(c) Let \( H(t,x) : \{0,1\} \times \varnothing \cap D(A) \to \mathbb{R}^n \), if \( H(t,x) \) is uniformly continuous in \( x \) for each \( t \), and for each \( t \in \{0,1\} \). \( H(t,.) \) is a strict \( k \)-set contraction, \( k \) doesn't depend on \( t \), suppose

\[
x \notin -A + H(t,x), \forall x \in \mathcal{D}(A), t \in \{0,1\}
\]

then \( i((A+K), \mathcal{D}(A)) \) doesn't depend on \( t \).

(d) If \( i((A+K), \mathcal{D}(A)) = 0 \), then \( x \in -A + Kx \) has a solution in \( \mathcal{D}(A) \), i.e \( -A + K \) has a fixed point.

PROOF: by the definition, (b), (c), (d) is obvious. (see(1)or (8))

Now, we prove (a). First, we have

\[
0 \in D(A), 0 \in A0 \quad (2.2)
\]

In fact, \( (A+I) D(A) = P \), so there exists \( x \in D(A), a \in A + Kx \), such that \( x + a = 0 \).

Since \( x \geq 0, a \geq 0 \), So we must have \( x + a = 0 \). Hence

\[
(A+I) - 0 = 0 \quad (2.3)
\]

by the definition, we need to prove

\[
i((I+A) - K, \mathcal{D}(A)) = 1, \mathcal{O} = B(0,r)
\]

(2.4)

So \( i((A+I) - K, \mathcal{D}(A)) = 1 \). In the following, \( K, A, \varnothing \) are same as above.

LEMMA 1: If \( Kx \in \varnothing, \forall x \in \mathcal{D}(A) \), then

\[
i((A+K), \mathcal{D}(A)) = 1
\]

PROOF: Let \( H(t,x) = tKx, t \in \{0,1\}, x \in \mathcal{D}(A) \). If \( x \in -A + tKx \) for some \( x \in \mathcal{D}(A) \) and \( t \in \{0,1\} \).
then \( t \neq 0 \) (otherwise, we get \( t = 0 \in \mathcal{O} \), a contradiction)

So \( K_t \supseteq -x \), a contradiction to \( Kx \ni x \).

Hence, \( H(t, x) \) satisfy all the conditions of (c) in theorem 1.

So

\[
\tau (-A + K, \mathcal{O} \cap D(A)) = \tau (-A + 0, \mathcal{O} \cap D(A))
\]

by (2, 3), we have \((I + A)^{-1} = 0 \in \mathcal{O} \cap P\)

So \( \tau (I + A)^{-1} = 0, \mathcal{O} \cap P = 1 \), and we get

\[
\tau (-A - 0, \mathcal{O} \cap D(A)) = 1
\]

Hence

\[
\tau (-A + K, \mathcal{O} \cap D(A)) = 1
\]

**COROLLARY 1**: If \( 0 \in \mathcal{O} \), and \( Kx \ni x \), then \(-A + K\) has a fixed point in \( \mathcal{O} \cap D(A) \)

**PROOF**: It’s obvious \( Kx \ni x \), \( x \in \mathcal{O} \cap P \). By lemma 1,

\[
\tau (-A + K, \mathcal{O} \cap D(A)) = 1
\]

Theorem 1, (d) implies \(-A + K\) has a fixed point in \( \mathcal{O} \cap D(A) \).

**LEMMA 2**: Let \( u_0 \neq 0 \), suppose \( x - tu_0, -A(x - tu_0) + Kx = 0 \), \( x \in \mathcal{O} \cap P \), and \( x - tu_0 \in D(A) \), for \( t \geq 0 \); Then

\[
\tau (-A + K, \mathcal{O} \cap D(A)) = 0
\]

**PROOF**: Suppose \( \tau (I + A)^{-1} K = 0 \).

For each \( t > 0 \), let \( H(t, x) = (I + A)^{-1} K + tu_0 \), \( x \in \mathcal{O} \cap P, t \in (0, 1) \); 

It’s obvious \( H(t, x) \) is uniformly continuous in \( x \) for each \( t \), and \( H(t, \cdot) \) is strict \( k \)-set contraction for each \( t \).

By (1), (see also (8)). We get

\[
\tau ((I + A)^{-1} K + tu_0, \mathcal{O} \cap P) = \tau (I + A)^{-1} K, \mathcal{O} \cap P = 0
\]

So there exists \( x \), \( x \in \mathcal{O} \cap P \), such that

\[
x = (I + A)^{-1} K x = tu_0
\]

Letting \( t \to \infty \), the left side of (2.6) is bounded, but the right side of (2.6) is unbounded, a contradiction.

We must have \( \tau (-A + K, \mathcal{O} \cap D(A)) = 0 \).

**THEOREM 2**: If \( A, D(A) \subseteq P \to \mathbb{R}^2 \) is an accretive mapping, \((I + A)D(A) = P, \mathcal{O}_1, \mathcal{O}_2\) are two open bounded subsets of \( E \), \( 0 \in \mathcal{O}_1 \subseteq \mathcal{O}_1, K, \mathcal{O} \cap P \), is a strict \( k \)-set contraction mapping, \( 0 \neq u_0 \in P \)

(i) For each \( x \in \mathcal{O}_1, x \notin Kx \), for each

\[
x \in \mathcal{O}_1 \cap P, x - tu_0 \in D(A), t \geq 0, x - tu_0, -A(x - tu_0) + Kx
\]

(ii) For each \( x \in \mathcal{O}_1, x \notin Kx \), for each

\[
x \in \mathcal{O}_1 \cap P, x - tu_0 \in D(A), t \geq 0, x - tu_0, -A(x - tu_0) + Kx
\]

Suppose either (i) or (ii) is satisfied, then \(-A + K\) has a fixed point in \((\mathcal{O}_1 - \mathcal{O}_2) \cap D(A)\)

**PROOF**: Suppose condition (i) is satisfied by, Lemma 1, we have

\[
\tau (-A + K, \mathcal{O}_1) = 1
\]

by Lemma 2, we have

\[
\tau (-A + K, \mathcal{O}_1) = 1
\]

by (b) of Theorem 1, and (6), (7). We get

\[
\tau (-A + K, \mathcal{O}_1 - \mathcal{O}_2) = 1
\]

by (d) of Theorem 1, we know \(-A + K\) has a fixed point in \((\mathcal{O}_1 - \mathcal{O}_2) \cap D(A)\).

If (ii) is satisfied, the proof is similar. We complete the proof.

**THEOREM 3**: For each \( x \in \mathcal{O} \cap D(A), \| Kx \| \leq \| x \|, \) and \( 0 \in \mathcal{O}_1 \), then \(-A + K\) has a fixed point in \( \mathcal{O} \cap D(A) \)

**PROOF**: We may suppose

\[
x \notin A x + K x, \mathcal{O} \subseteq D(A)
\]

Let \( H(t, x) = K x \), \( x \in \mathcal{O} \cap P, t \in (0, 1) \); 

It’s obvious \( H(t, x) \) is uniformly continuous in \( x \) and \( H(t, \cdot) \) is strict \( k \)-set contraction for each \( t \).

We show that

\[
x \notin A x + H(t, x), x \in \mathcal{O} \cap D(A), t \in (0, 1)
\]

If \( x \in A x + H(t, x) \) for some \( x \in \mathcal{O} \cap D(A), t \in (0, 1) \), then \( x = (I + A)^{-1} H(t, x) \).
Since \((I+A)^{\dagger}\) is nonexpansive and \((I+A)^{\dagger} 0=0.\) So
\[
\| x \| \leq \| H(t,x) \| = \| tKx \| \leq t \| x \|
\]
Therefore \(t=1,\) contradict to (8) by (c) of Theorem 1.

\[
(1-A+K, D(A)) = t(1-A+K, D(A))
\]
and (2.5) implies \(t(-A+K, D(A)) = 1.\)

by (d) of Theorem 1, \(-A+K\) has a fixed point in \(D(A).\)

**THEOREM 4:** If \(0,\ D(A).\)

**PROOF:** We may assume \(x \in \cdot A x + K x, x \in D(A).\)

Let \(H(t,x) = tKx, t \in [0,1].\)

If \(x \in -A x + t K x\) for some \(t \in [0,1], x \in D(A),\) then \(tKx \in x + A x\)

So there exists \(a \in A x,\) such that \(tKx = x + a.\) We have \(\| K x \| \leq t \| K x \|\)

By the assumption (2.11), \(t \neq 1,\) we must have \(K x = 0, x + a = 0\)

So we have \(x \in -A x + H(t,x), x \in D(A), t \in [0,1].\)

The following proof is similar to that of Theorem 3. This end the proof.

**REFERENCES**


10. P. M. FITZPATRICK, W. V. PETRYSHYN. Fixed Point theorems and the fixed point index for multivalued mappings in cones. J. L. Math. Soc. 12(1975)75–85


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