ON APPROXIMATION OF FUNCTIONS AND THEIR DERIVATIVES BY QUASI-HERMITE INTERPOLATION

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ABSTRACT. In this paper, we consider the simultaneous approximation of the derivatives of the functions by the corresponding derivatives of quasi-Hermite interpolation based on the zeros of \((1 - x^2)p_n(x)\) (where \(p_n(x)\) is a Legendre polynomial). The corresponding approximation degrees are given. It is shown that this matrix of nodes is almost optimal.

KEY WORDS: Hermite interpolation, optimal nodes, derivatives, Legendre polynomials, best approximation.


1 INTRODUCTION.

Let

\[-1 \leq x_n < \ldots < x_1 < x_0 \leq 1\]  

be an arbitrary nodes system on \([-1, 1]\) and let \(f \in C^1[-1, 1]\). We consider the Hermite interpolation operator:

\[H_n(f, x) := \sum_{k=0}^{n} f(x_k)l_k(x) + \sum_{k=0}^{n} f'(x_k)\sigma_k(x),\]  

where

\[l_k(x) = v_k(x)\ell_k^2(x), \quad \sigma_k(x) = (x - x_k)^2\ell_k^2(x),\]

\[l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)},\]

\[v_k(x) = 1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k), \quad \omega(x) = \prod_{k=0}^{n}(x - x_k).\]

It satisfies the following conditions:

\[H_n(f, x_k) = f(x_k), \quad k = 0, 1, \ldots, n\]

and

\[H'_n(f, x_k) = f'(x_k), \quad k = 0, 1, \ldots, n\]

There have been many articles considering the problem of approximation to \(f(x)\) by \(H_n(f, x)\). Generally, we consider approximation of \(f'(x)\) by the derivative of Hermite interpolation. We know that the convergence

\[\lim_{n \to \infty} ||H'_n(f, x) - f'(x)|| = 0,\]

does not hold for all \(f \in C^1[-1, 1]\) (here ||.|| is the maximum norm). Pottinger [1] investigated this problem when \(\{x_k\}_{k=0}^{n}\) are the zeros of the Tchebycheff polynomial of the first kind and obtained the following result:
\[ ||H_n^r(f, x) - f'(x)|| = O(n)E_{2n}(f'), \quad (1.3) \]

where \( E_n(f) \) is the best approximation of \( f(x) \). (The factor \( O(n) \) is best possible, cf. Steinhaus [2].) In [3], Szabados and Varma introduced a norm for the higher derivatives of the operator (1.2):

\[ ||H_n^r|| = \sup \{ ||H_n^r(f, x)|| : |f^{(k)}(x_k)| \leq n'(1 - x_k^2)^{-r/2}, k = 1, \ldots, n; i = 0, 1 \} \]

\((r, n = 1, 2, \ldots)\) and they proved that for any system of nodes ([3, Theorem 1])

\[ ||H_n^r|| \geq c_n n^r \ln n, \quad (n, r = 1, 2, \ldots) \quad (1.4) \]

where \( c_n > 0 \) depends only on \( r \). Moreover, for the matrix of nodes:

\[ \omega(x) = P_{n-2t+1}(x)H_j^1(x^2 - \cos^2 \frac{(j-1)\pi}{3(t(n-2t+1))}, \quad (1.5) \]

they obtain ([3, Theorem 3])

\[ ||H_n^r|| = O(n^r \ln n), \quad (1.6) \]

where \( t = \lceil \frac{r+1}{4} \rceil, \alpha = 2t - \frac{r+1}{4} \) \((r \geq 1 \text{ integer})\) and \( P_{n-2t+1}(x) \) are the ultraspherical Jacobi polynomials of degree \( n-2t \). Moreover, \( \alpha \) takes only the values \(-1/2, 0, 1/2, 1\) according to \( r = 0, 3, 2, 1 \mod 4 \). (See [3, Remark, P305].) Therefore for the matrix of nodes defined by (1.5) we have

\[ ||H_n^r(f, x) - f'(x)|| = O(\ln n)\omega(f^{(r)}, \frac{1}{n}), \quad (1.7) \]

(see [3]) At the end of paper [3], they speculated that "it would be interesting to construct a matrix which is optimal for all the derivatives up to order \( r \)." This is the problem of constructing matrix nodes so that the corresponding simultaneous approximation of \( f(x) \) from the first derivative to the \( r \)-th derivative is optimal by the corresponding Hermite interpolation.

Remark: With respect to Lagrange interpolation, the complete solution of minimizing the corresponding derivatives norm to (1.4) was given by Szabados [4] (also see Vértesi [5]). The main idea is that adding nodes (near \( x=\pm 1 \)) to Jacobi nodes make the similar estimates of (1.4) optimal.

In this paper, we point out that for the quasi-Hermite interpolation \( R_n(f, x) \) based on the zeros of \((1 - x^2)p_n(x) \) (where \( p_n(x) \) is the Legendre polynomial with normalization: \( p_n(1) = 1 \)), we have

**THEOREM 1.** If \( f \in C^1[-1,1] \), then

\[ ||R_n^r(f, x) - f'(x)|| = O(\ln n)E_{2n}(f'). \quad (1.8) \]

**THEOREM 2.** If \( f \in C^r[-1,1] \) \((r \geq 2)\), then

\[ ||R_n^r(f, x) - f'(x)|| = O(\ln n)E_{2n}(f') = O(\frac{\ln n}{n})E_{2n-1}(f''), \quad (1.9) \]

\[ ||\sqrt{1 - x^2}(R_n^r(f, x) - f''(x))|| = O(\ln n)E_{2n-1}(f''), \quad (1.10) \]

and

\[ ||R_n^{(i)}(f, x) - f^{(i)}(x)||_{-\sigma, \sigma} = O(\ln n)E_{2n-i+1}(f^{(i)}), \quad i = 2, \ldots, r \quad (1.11) \]

where \( 0 < \sigma < 1 \).

From this we see that the zeros of \((1 - x^2)p_n(x) \) are almost optimal and the corresponding simultaneous approximation is better than that of Hermite interpolation based on the zeros of Tchebyshev polynomial of the first kind.

Remark: We conjecture that the factor \( \sqrt{1 - x^2} \) in (1.10) cannot be removed on the whole interval \([-1,1] \), in which case the preceding results are optimal.
2 LEMMAS.

In order to prove the Theorems, we state some properties of Legendre polynomials (see Szegö [6]).

\[ |p_n(x)| \leq 1, \quad (2.1) \]
\[ (1 - x^2)^{1/4}|p_n(x)| \leq (2/\pi n)^{-1/2}, \quad n \geq 2 \quad (2.2) \]
\[ (1 - x^2)^{3/4}|p'_n(x)| \leq (2n)^{1/2}, \quad n \geq 3 \quad (2.3) \]
\[ \sin^2 \theta_k = 1 - z_k^2 > (k - 3/2)^2 n^{-2}, \quad k = 1, ..., \lfloor n/2 \rfloor \quad (2.4) \]
\[ |p'_n(z_k)| > c(k - 3/2)^{-3/2} n^2, \quad k = 1, ..., \lfloor n/2 \rfloor \quad (2.5) \]

We note that in (2.4) and (2.5) similar estimates hold for \( k = \lfloor n/2 \rfloor, ..., n \). On combining (2.4) and (2.5), it follows that

\[ \left[ (1 - z_k^2)^{3/4}|p'_n(z_k)| \right]^2 \geq cn, \quad k = 1, ..., n \quad (2.6) \]

where \( c \) is an absolute positive constant independent of \( f \) and \( n \), whose value may vary from line to line throughout our paper.

Let

\[-1 = x_{n+1} < x_n < ... < x_1 < x_0 = 1\]

be the zeros of \((1 - x^2)p_n(x)\). Then its corresponding quasi-Lermete interpolation is the following

\[ R_n(f, x) = \sum_{k=0}^{n+1} f(x_k)r_k(x) + \sum_{k=1}^{n} f'(x_k)\gamma_k(x), \quad (2.7) \]

where

\[ r_0(x) = \frac{1 + x}{2} p'_n(x), \quad r_{n+1} = \frac{1 - x}{2} p'_n(x), \]
\[ r_k(x) = \frac{1 - x^2}{1 - z_k^2} l_k(x), \quad k = 1, ..., n \]
\[ \gamma_k(x) = (x - x_k)r_k(x), \quad k = 1, ..., n \]
\[ l_k(x) = \frac{p_n(x)}{p'_n(x_k)(x - x_k)}, \quad k = 1, ..., n \]

It satisfies that

\[ R_n(f, x_k) = f(x_k), \quad k = 0, 1, ..., n + 1. \]

and

\[ R_n'(f, x_k) = f'(x_k), \quad k = 1, ..., n \]

**LEMMA 1.** We have

\[ \sqrt{1 - x_k^2} \leq \sqrt{1 - x^2} + 2\frac{|x - x_k|}{\sqrt{1 - x^2}}, \quad k = 1, ..., n. \]

**PROOF.** One easily sees that

\[ \sqrt{1 - x_k^2} = \sqrt{1 - x^2} + \sqrt{1 - x_k^2} - \sqrt{1 - x^2} \]
\[ = \sqrt{1 - x^2} + \frac{x^2 - x_k^2}{\sqrt{1 - x_k^2} + \sqrt{1 - x^2}} \leq \sqrt{1 - x^2} + 2\frac{|x - x_k|}{\sqrt{1 - x_k^2}} \]

This proves Lemma 1. \( \Box \)
LEMMA 2. We have

\[(i) \quad I_1 := \sum_{k=1}^{n} \frac{|x - x_k|}{1 - x_k^2} l^2_k(x) = O(\ln n) \quad (2.8)\]

\[(ii) \quad I_2 := \sum_{k=1}^{n} |x - x_k| \frac{1 - x^2}{1 - x_k^2} |l_k(x)'(x)| = O(\ln n) \quad (2.9)\]

PROOF. From Lemma 1 we have

\[I_1 \leq \sum_{k=1}^{n} \frac{\sqrt{1 - x^2} |x - x_k|}{(1 - x_k^2)^{3/2}} l^2_k(x) + 2 \sum_{k=1}^{n} \frac{|x - x_k|^2}{(1 - x_k^2)^{3/2}} l^2_k(x) := A_1(x) + A_2(x) \quad (2.10)\]

Throughout this paper we assume \(x_i\) to be the zero of \(p_n(x)\) which is the nearest to \(x\) and \(i = |k - j|\).

By using (5.8) in Prasad and Varma[7] we have

\[\sqrt{1 - x^2} \frac{|x - x_j|}{1 - x_j^2} l^2_j(x) \leq \frac{c}{n}. \quad (2.11)\]

Notice that, with \(x = \cos \theta \quad (0 \leq \theta \leq \pi)\)

\[\sin \theta \leq \sin \theta + \sin \theta_k \leq 2 \sin \frac{\theta + \theta_k}{2}, \]

so we have

\[A_1(x) = \frac{1}{\sqrt{1 - x_j^2}} \frac{\sqrt{1 - x^2} |x - x_j|}{1 - x_j^2} l^2_j(x) + \sum_{k \neq j} \frac{\sqrt{1 - x^2} |x - x_k|}{(1 - x_k^2)^{3/2}} l^2_k(x) \leq \frac{c}{1 - \sin \theta_j} + \sum_{k \neq j} \frac{1}{|1 - x_k^2)^{3/4}| |p_n(x_k)|} |x - x_k| \]

\[= O(1)[1 + p_n^2(x)] \sum_{k \neq j} \frac{1}{\sin |x - x_k|} = O(1)[1 + p_n^2(x)] \sum_{k \neq j} \frac{n}{|x - x_k|} = O(\ln n). \]

Similarly,

\[A_2(x) = \sum_{k=1}^{n} \frac{p_n^2(x)}{[(1 - x_k^2)^{3/4}|p_n(x_k)|]} \frac{1}{\sqrt{1 - x_k^2}} = O(1)\]

\[p_n^2(x) \sum_{k=1}^{n} \frac{1}{\sqrt{1 - x_k^2}} = O(\ln n), \]

so we obtain (2.8).

Notice that

\[l_k'(x) = \frac{p_n'(x)(x - x_k) - p_n(x)}{(x - x_k)^2 p_n^2(x_k)}, \]

and we have

\[I_2 \leq \sum_{k=1}^{n} |x - x_k| \frac{(1 - x^2)|x - x_k||p_n'(x)|}{(1 - x_k^2)^{3/4}|p_n(x_k)|} l_k(x) + \sum_{k=1}^{n} r_k(x) := B_1(x) + B_2(x) \]

One notes Prasad and Varma[7]

\[\frac{(1 - x^2)^{3/4}}{p_n(x_k)^{3/4}} |l_k(x)| \leq c, \]

so we have

\[B_1(x) = \sum_{k=1}^{n} \frac{(1 - x^2)^{3/4}|p_n'(x)|}{(1 - x_k^2)^{3/4}|p_n(x_k)|} \frac{(1 - x^2)^{3/4}}{p_n(x_k)^{3/4}} |l_k(x)| \]

\[= O(1)\frac{(1 - x^2)^{3/4}|p_n'(x)|}{(1 - x_k^2)^{3/4}|p_n(x_k)|} \frac{(1 - x^2)^{3/4}}{p_n(x_k)^{3/4}} |l_k(x)| \]

\[= O(1)[1 + \frac{1}{n} \frac{(1 - x^2)|p_n(x)p_n'(x)|}{\sin \frac{\theta_k}{|x - x_k|}} \sum_{k \neq j} \frac{\sin \theta_k}{|x - x_k|} \]

\[= O(1)[1 + \ln n(1 - x^2)|p_n(x)p_n'(x)|] = O(\ln n). \]
Obviously,  
\[ B_2(x) \leq \sum_{k=0}^{n+1} r_k(x) = 1. \]

Therefore we obtain (2.9). \( \square \)

**Lemma 3.** We have  
\[ I_3 := \sum_{k=0}^{n+1} (1 - x^2)^k r_k(x) \cdot \] 
\[ \text{and} \] 
\[ I_4 := \sum_{k=1}^{n+1} \sqrt{1 - x^2} |\gamma_k(x)| = O\left(\frac{\ln n}{n}\right)\sqrt{1 - x^2}. \] 

**Proof.** Since  
\[ I_3 = \sum_{k=1}^{n+1} I_k^2(x), \]

from Nevai and Vértesi [8] we have  
\[ \sum_{k=1}^{n+1} I_k^2(x) = O(1)(1 + J_n^2(x) + \frac{\ln n}{n} J_n^2(x)), \]

where \( J_n(x) \) is the orthonormal Legendre polynomials:  
\[ \int_{-1}^{1} J_n(x) J_m(x) \, dx = \delta_{nm}, \]

and notice that Natanson [9] gives  
\[ ||J_n(x)|| = O(1)n^{1/2}. \]

It follows that  
\[ \sum_{k=1}^{n+1} I_k^2(x) = O(\ln n), \]

this implies (2.12). Also, we have  
\[ I_4 = \sum_{k=1}^{n+1} \frac{(1 - x^2)|z - z_k|}{\sqrt{1 - z_k^2}} I_k^2(x) \]

\[ = (1 - x^2) \frac{1}{(1 - z_k^2)^{1/2}} |\rho_n(x)| \frac{1}{(1 - z_k^2)^{3/4}} |\rho_n(x_k)| |f_j(x)| + \sum_{k \neq j} \frac{(1 - x^2) p_n^2(x)}{(1 - z_k^2)^{3/4} |\rho_n(x_k)|} \frac{1 - x^2}{|z - z_k|}. \]

Recall that (Erdős [10]) for \(-1 \leq x \leq 1,\)
\[ |l_k(x)| \leq 1, \quad k = 1, \ldots, n \]

therefore, similar to the estimates of \( I_1 \) and \( I_2, \) we have  
\[ I_4 = O(1) \frac{1 - x^2}{n} + \left(\frac{1 - x^2}{n}\right) p_n^2(x) \sum_{k \neq j} \frac{1}{\sin \left|\frac{\pi - \theta_k}{2}\right|} = O\left(\frac{\ln n}{n}\right)\sqrt{1 - x^2}. \]

This proves Lemma 3. \( \square \)

Remark: If we need not want to obtain the factor \((1 - x^2),\) we can obtain a better estimate of \( I_3. \)

**Lemma 4.** Let \( f \in C^r[-1, 1], \) then there exist polynomials \( q_n(x) \) of degree \( n \geq 4r + 5 \) such that \((j = 0, 1, \ldots, r)\)
\[ |f^{(j)}(x) - q_n^{(j)}(x)| = O(1) \left(\frac{\sqrt{1 - x^2}}{n}\right)^{r-j} E_{n-r}(f^{(r)}). \]  
(2.14)
PROOF. From Gopengz's Theorem [11] we know that there exist polynomials \( t_n(x) \) of degree \( n \geq 4r + 5 \) such that
\[
|f^{(j)}(x) - t_n^{(j)}(x)| \leq c\left(\frac{1 - x^2}{n}\right)^{r-j} \omega(f^{(r)}, \frac{1 - x^2}{n})
\]
Let \( s_n(x) \) be the polynomial of degree \( n > r \) such that
\[
\|f^{(r)}(x) - s_n^{(r)}(x)\| \leq E_{n-r}(f^{(r)}),
\]
then we have
\[
|f^{(j)}(x) - q_n^{(j)}(x)| := |f^{(j)}(x) - (s_n^{(j)}(x) + t_n^{(j)}(x))|
\]
\[
\leq c\left(\frac{1 - x^2}{n}\right)^{r-j} \omega((f - s_n)^{(r)}, \frac{1}{n}) = O(1)\left(\frac{1 - x^2}{n}\right)^{r-j}\|f^{(r)} - s_n^{(r)}\|
\]
\[
= O(1)\left(\frac{1 - x^2}{n}\right)^{r-j} E_{n-r}(f^{(r)}).
\]
This proves Lemma 4. \( \square \)

**LEMMA 5.** Let \( s_j(x) \) be a polynomial of degree \( \leq n \), and suppose that the inequality
\[
\sum_{j=1}^{m} |s_j(x)| = O(1), \quad -1 \leq x \leq 1.
\]
holds. Then
\[
(1 - x^2)^{n/2} \sum_{j=1}^{m} |s_j^{(i)}(x)| = O(1)n^i,
\]
where \( m \geq 1 \) and \( 1 \leq i \leq n \).

**PROOF.** Although Ramm [12, Lemma 1, p285] only proved the case of \( i=1 \), (26) can be obtained by using a completely similar method. \( \square \)

### 3 PROOFS OF THEOREMS.

**PROOF OF THEOREM 1.** Notice that
\[
R_n(f, x) - f(x) = \sum_{k=0}^{n+1} (f(x_k) - f(x))r_k(x) + \sum_{k=1}^{n} f'(x_k)\gamma_k(x)
\]
\[
= \sum_{k=0}^{n+1} \int_{x_k}^{x} f'(t) \, dt \gamma_k(x) + \sum_{k=1}^{n} f'(x_k)\gamma_k(x).
\]
This implies
\[
||R_n|| \leq \sum_{k=0}^{n+1} |x - x_k|r_k'(x) + \sum_{k=1}^{n} |\gamma_k'(x)| ||f'||
\]
(3.1)

One easily sees that
\[
(1 - x)|r_k'(x)| \leq (1 - x)|p_k^2(x)| + |y_k p_n^*(x)| = O(1).
\]
Similarly we have
\[
(1 + x)|r_{n+1}(x)| = O(1).
\]

Notice that
\[
r_k'(x) = \frac{2x}{1 - x_k^2} \gamma_k^2(x) + \frac{2(1 - x^2)}{1 - x_k^2} \rho_k(x) \gamma_k(x)
\]
and
\[
\gamma_k'(x) = r_k(x) + (x - x_k)r'_k(x).
\]
From Lemma 2 we have
and also we have
\[ \sum_{k=1}^{n} |\gamma_k'(x)| = O(\ln n). \] (3.3)

It now follows that
\[ ||R_n''|| = O(\ln n)||f'||. \] (3.4)

Combining Lemma 4, (3.2) and (3.3), we obtain Theorem 1. \[ \square \]

PROOF OF THEOREM 2. Theorem 1 implies (9). Here we only prove the case \( i = 2 \). The other cases are completely similar. By using Lemma 5 (or see Borwein and Erdelyi [13]) and from Lemma 3 we obtain the following
\[ \sum_{k=0}^{n+1} (1 - x_k^2) |r''_n(x)| = O(n^2 \ln n) \] (3.5)

and
\[ \sqrt{1 - x^2} \sum_{k=1}^{n} \sqrt{1 - x_k^2} |\gamma_k''(x)| = O(n \ln n) \] (3.6)

Notice that
\[ R_n''(f, x) - f''(x) = R_n''(f - q_{2n+1}, x) + q_{2n+1}'(x) - f''(x) \]
and
\[ R_n''(f - q_{2n+1}, x) = \sum_{k=0}^{n+1} (f(x_k) - q_{2n+1}(x_k)) r''_n(x) + \sum_{k=1}^{n} (f'(x_k) - q_{2n+1}'(x_k)) \gamma_k''(x). \]

Combining Lemma 4, (3.5) and (3.6), we obtain (1.10). \[ \square \]

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References


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