APPLICATION ON LOCAL DISCRETE EXPANSION

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ABSTRACT. The process of changing a topology by some types of its local discrete expansion preserves s-closeness, S-closeness, semi-compactness, semi-$T_i$, semi-$R_i$, $i \in \{0,1,2\}$, and extremely disconnectness Via some other forms of such above replacements one can have topologies which satisfy separation axioms the original topology does not have

KEY WORDS AND PHRASES: Near open sets, local discrete expansion, extremely disconnected, semi-compact, s-closed, S-closed, semi-$T_i$, semi-$R_i$, and cid spaces

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1. INTRODUCTION

Throughout the present paper $(X, \tau)$ is a topological space (or simply a space $X$) on which no separation axioms are assumed unless explicitly stated. For any $B \subseteq X$, $\text{cl}_\tau B$ (resp. $\text{int}_\tau B$) denotes the closure (resp. interior) of $B$. A subset $B$ is said to be regular open (resp. regular closed) if $B = \text{int}_\tau (\text{cl}_\tau B)$ (resp. $B = \text{cl}_\tau (\text{int}_\tau B)$). A subset $B$ of a space $X$ is said to be $\tau$-semi open [12] (resp $\tau$-regular semi-open [2]) if there exists a $\tau$-open (resp. $\tau$-regular open) set $U$ satisfying $U \subseteq B \subseteq \text{cl}_\tau U$. $B$ is $\tau$-semi-closed [3] if the set $X - B$ is $\tau$-open. The family of all regular open (resp. regular semi-open, semi-open) sets in $X$ is denoted by $\text{RO}(X, \tau)$ (resp $\text{RSO}(X, \tau), \text{SO}(X, \tau)$). The union (resp. intersection) of all $\tau$-semi-open (resp. $\tau$-semi-closed) sets contained in $B$ (resp. containing $B$) is called the $\tau$-semi-interior [3] (resp $\tau$-semi-closure [3]) of $B$, and it is denoted as $s\text{-int}_\tau B$ (resp. $s\text{-cl}_\tau B$). A space $X$ is said to be extremely disconnected (denoted by E.D) if for every open set $U$ of $X$, $\text{cl}_\tau U$ is open in $\tau$.

The concept of local discrete expansion of a topology was first introduced by S.P. Young in 1977 [17], "Let $(X, \tau)$ be a topological space and $A$ be any subset of $X$. The topology $\tau[A] = \{U - H : U \in \tau, H \subset A\}$ is called the local discrete expansion of $\tau$ by $A$. A space $X$ is semi-$T_2$ [13] (resp. semi-$T_\Delta$ [11]) iff for $x, y \in X$, $x \neq y$ there exist $U$ and $V \in \text{SO}(X, \tau)$, $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$ (resp $\text{cl}_\tau U \cap \text{cl}_\tau V = \emptyset$). Semi-$T_0$ and semi-$T_1$ were introduced to topological spaces [13] by replacing the word "open" by "semi-open" in the definitions of $T_0$ and $T_1$ respectively. A space $X$ is semi-$R_0$ [6] iff for each semi-open set $U$ and $x \in U$, $s - \text{cl}_\tau \{x\} \subset U$. A space $X$ is semi-$R_1$ [6] iff for $x, y \in X$ such that $s - \text{cl}_\tau \{x\} \neq s - \text{cl}_\tau \{y\}$ there exist disjoint semi-open sets $U$ and $V$ such that $s - \text{cl}_\tau \{x\} \subset U$, and $s - \text{cl}_\tau \{y\} \subset V$. A space $X$ is called cid [15] if every countable infinite subspace of $X$ is discrete. A space $X$ is semi-compact [7] (resp. s-closed [5], S-closed [16]) if for every cover $\{V_i : i \in I\}$ of $X$ by semi-open sets of $X$, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{V_i : i \in I_0\}$ (resp. $X = \bigcup \text{sc}(V_i) : i \in I_0\}$, $X = \bigcup \text{cl}(V_i) : i \in I_0\}$).

REMARK 1.1. For a subset $A$ of a space $(X, \tau)$ we say that $A$ satisfies condition $(C_1)$ if $A \cup U = \emptyset$, for every $U \in \tau - \{X\}$.

Listed below are theorems that will be utilized in this paper.

THEOREM 1.1 [14] If $\tau$ and $\tau'$ are two topologies on $X$ such that $\tau \subset \tau'$, then $\text{RO}(X, \tau') = \text{RO}(X, \tau')$ iff $\text{cl}_\tau G = \text{cl}_\tau G$ for every $G \in \tau'$ [equivalent iff $\text{int}_\tau F = \text{int}_\tau F$, for every $F \in \tau'$]

THEOREM 1.2 [11] If $X$ is a space, and $A \subset X$ satisfying $(C_1)$ then, $\text{cl}_\tau A \subset G = \text{cl}_\tau G$, for every $G \in \tau[A]$
THEOREM 1.3 [4] If \( X \) is a space, and \( A \in SO(X, \tau) \) such that \( A \subset B \subset cl_A A \) Then, \( B \in SO(X, \tau) \)

THEOREM 1.4 [10] If \( X \) is a space, and \( B \subset X \), then \( s - cl, B = B \cup int, cl, B \)

THEOREM 1.5 [8] A space \( X \) is \( E D \) iff for every pair \( U \) and \( V \) of disjoint \( \tau \)-open sets, we have \( cl_U \cap cl_V = \phi \)

THEOREM 1.6 [5] A space \( X \) is \( s \)-closed iff every cover of \( X \) by regular semi-open sets has a finite subcover

THEOREM 1.7 [15] (a) A space \( X \) is cid if every countable infinite subset is closed
(b) Any infinite cid space is \( T_1 \)

THEOREM 1.8 [17] Let \( A \) be any subset of \( X \) Then \( (A, \tau[A] \cap A) \) is discrete

THEOREM 1.9 [17] Let \( A \) be a closed subset of \( X \) Then \( (A, \tau \cap A) \) is a discrete subspace of \( X \) iff \( \tau = \tau[A] \)

THEOREM 1.10 [9] Let \( X \) be a \( T_1 \)-space Then \( X \) is cid iff countable subsets have no limits points

2. ON LOCAL DISCRETE EXPANSION

THEOREM 2.1. If \( (X, \tau) \) is a space and \( A \subset X \), then
(i) \( SO(X, \tau[A]) \subset \{B - H : B \in SO(X, \tau), H \subset A\} \)
(ii) If \( A \) satisfying \( (C_1) \), then the inclusion symbol in (i) is replaced by equality

PROOF. (i) Let \( W \in SO(X, \tau[A]) \), then there exists \( V \in \tau[A] \) such that \( V \subset W \subset cl_{\tau[A]}V \)
Then \( (U - H_1) \subset W \subset cl_{\tau[A]}(U - H_1) \), where \( U \in \tau, H_1 \subset A \) Put \( H_2 = U \cap H_1 \), then \( H_2 \subset A \), and
\( (U - H_1) \cup H_2 \subset W \cup H_2 \subset cl_{\tau[A]}(U - H_1) \cup H_2 \) Then \( U \subset W \cup H_2 \subset cl_{\tau[A]}U \subset cl_U, \)
and \( (W \cup H_2) \in SO(X, \tau) \) Put \( B = W \cup H_2 \), and \( H = H_1 - W \subset A \) Then \( B - H = W \cup (U \cap H_1) - (H_1 - W) = W \).

(ii) By Theorem 1.2, the proof is obvious

REMARK 2.1. From Theorem 2.1, it is easy to prove that, for any \( A \subset X \)
\( SO(X, \tau) \subset SO(X, \tau[A]) \)

THEOREM 2.2. If \( (X, \tau) \) is a space, and \( A \subset X \) satisfying \( (C_1) \) Then
(i) \( SO(X, \tau) = SO(X, \tau[A]) \).
(ii) \( RSO(X, \tau) = RSO(X, \tau[A]) \).

PROOF. In general \( SO(X, \tau) \subset SO(X, \tau[A]) \). To prove the converse, let \( W \in SO(X, \tau[A]) \), then there exists \( V \in \tau[A] \) satisfying \( V \subset W \subset cl_{\tau[A]}V \). Then \( (U - H) \subset W \subset cl_{\tau[A]}(U - H), \)
\( U \in \tau, H \subset A \). There are two cases.
(a) \( U \neq X \), then \( U - H = U \) Since \( cl_{\tau[A]}U = cl_U, U \), then \( W \in SO(X, \tau) \).
(b) \( U = X \), then \( (X - H) \subset W \subset cl_{\tau[A]}(X - H), cl_{\tau[A]}(X - H) \subset cl_{\tau[A]}(X - H) \), then \( cl_A \subset (X - U), cl_{\tau[A]} \cup U = \phi \), implies to \( cl_A \subset U \), for each \( U \in \tau \) \( \subset \{X\} \) Hence \( U \notin cl_A, cl_A \), and \( int, cl_A = \phi \), and \( H \) is a \( \tau \)-semi-closed set Thus \( (X - H) \in SO(X, \tau) \) From
Theorem 1.3, \( W \in SO(X, \tau) \)

(ii) By Theorems 1.1 and 1.2, the proof is obvious

COROLLARY 2.1. If \( X \) is a space, and \( A \subset X \) satisfying \( (C_1) \) Then
(i) \( (X, \tau) \) is semi-T, iff \( (X, \tau[A]) \) is semi-T, \( \{0, 1, 2, 3\} \).
(ii) If \( (X, \tau) \) is semi-T2, then \( (X, \tau[A]) \) is semi-T2.
(iii) If \( (X, \tau) \) is semi-R, then \( (X, \tau[A]) \) is semi-R, \( \{0, 1, 2\} \)

PROOF. By Theorem (2 2), the proof is obvious

THEOREM 2.3. If \( X \) is a space, and \( A \subset X \) satisfying \( (C_1) \). Then \( s - cl_{\tau[A]}G = s - cl, G \), for every \( G \in \tau[A] \)

PROOF. Let \( G \in \tau[A] \), then \( s - cl_{\tau[A]}G = G \cup int_{\tau[A]}cl_{\tau[A]}G = G \cup int_{\tau[A]}G = G \cup int_{\tau[A]}G = s - cl, G \) [by Theorems 1 1, 1 2 and 1 4]
THEOREM 2.4. If \( X \) is a space, and \( A \subset X \) satisfying (C1). Then \( (X, \tau) \) is E.D. iff \( (X, \tau[A]) \) is E.D.

PROOF. Let \( (X, \tau) \) be E.D., \( W \subset \tau[A] \). Then \( W = U - H, U \in \tau, H \subset A \).

But \( cl_{\tau[A]}(U - H) = cl_{\tau[A]}U = cl\tau U, \) and \( cl\tau U \subset \tau \). Thus \( \tau[A] \subset \tau \), and \( (X, \tau[A]) \) is E.D. Conversely, let \( (X, \tau[A]) \) be E.D., and \( U, V \in \tau \) such that \( cl\tau U \cap cl\tau V \neq \emptyset \). By Theorem 1.2, \( cl\tau U \cap cl\tau V \neq \emptyset \) if \( U \cap V \neq \emptyset \) by [Theorem 1.5]. Hence \( (X, \tau) \) is E.D.

THEOREM 2.5. If \( X \) is a space, and \( A \subset X \) satisfying (C1). Then \( (X, \tau) \) is semi-compact (resp. s-closed) iff \( (X, \tau[A]) \) is semi-compact (resp. s-closed).

PROOF. By Theorem 2.2, the proof is obvious.

THEOREM 2.6. If \( X \) is a space, and \( A \subset X \), and \( (X, \tau[A]) \) is S-closed (resp. s-closed), then \( (X, \tau) \) is S-closed (resp. s-closed).

PROOF. Since \( SO(X, \tau) \subset SO(X, \tau[A]) \), the proof is obvious.

3. L - \( T_i \) AND Q - L - \( T_i \) SPACES

Let \( R \) be a topological property which is preserved under expansions

DEFINITION 3.1. A topological space \( (X, \tau) \) is called L- \( R \) if there exists a subset \( S \subset X \) and \( S \neq X \), such that \( (X, \tau[S]) \) has \( R \).

PROPOSITION 3.1. If \( \tau \subset \tau' \), then for any \( S \subset X, \tau[S] \subset \tau'[S] \).

REMARK 3.1. If \( \tau \subset \tau' \) and \( \tau \) is L \( R \), then \( \tau' \) is also L \( R \), i.e. any expansion of L \( R \) topology on \( X \) is also L \( R \).

DEFINITION 3.2. Let \( \tau, \sigma, \tau', \sigma' \text{ and } j \in \{0, 1, 2, 2.5\} \). We say that \( (X, \tau) \) is Q \( L \) \( T_j \), if it is L \( T_j \) and T \( j \) \( \text{ and } j < i \).

Now we are going to show that some of the properties L \( T_i \) and Q \( L \) \( T_i \) are satisfied for some spaces but not for some other spaces.

PROPOSITION 3.2. For a space \( X \), the following diagram is easily obtained.

\[ T_{2.5} \Rightarrow Q - L - T_2 \Rightarrow T_2 \Rightarrow Q - L - T_2 \Rightarrow T_1 \Rightarrow Q - L - T_1 \Rightarrow T_0. \]

EXAMPLE 3.1. Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, X, \{a, b\}, \{c, d\}\} \) is not \( T_0 \) if \( A = \{a, c\} \), then \( \tau[A] = \{\phi, X, \{b\}, \{d\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, b, d\}\} \) is \( T_0 \). This example is \( Q - L - T_0 \).

The following is an example of a Q - L - T \( 2.5 \) but not \( T_{2.5} \).

EXAMPLE 3.2. Let \( X = N \times Z \cup \{(1, 0), (-1, -1)\} \) where \( N \) is the natural numbers and \( Z \) the integers. The topology has as its base sets of the following forms:

\[ \{(m, n)\} \quad n \neq 0, \quad m \neq -1 \]
\[ U_n((a, 0)) = \{(a, 0)\} \cup \{(m, n)\} \mid m \geq n, \quad n \in N \]
\[ U_n((-1, 1)) = \{(-1, 1)\} \cup \{(a, m)\} \mid a \geq n, m > 0, \quad n \in N \]
\[ U_n((-1, -1)) = \{(-1, -1)\} \cup \{(a, m)\} \mid a \geq n, m < 0, \quad n \in N \]

This space is \( T_2 \) but not \( T_{2.5} \) as \((-1, 1) \) and \((-1, -1) \) do not have disjoint closed neighborhoods. Choosing \( A = N \times (Z \setminus \{0\}) \), the discrete expansion is the discrete topology and thus \( T_2 \).

EXAMPLE 3.3. Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, X, \{b\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}\} \), then \( \tau[A] = \text{Discrete} \). This example is \( Q - L - T_1 \) but not \( T_1 \) and is an example of a space which is not \( Q - L - T_2 \).

EXAMPLE 3.4. Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a, b\}\} \). If \( A = \{a, b\} \), then \( \tau[A] = \text{Discrete} \). This example is not \( Q - L - T_1 \).

The excluded point topology on an infinite set \( X \) is the family consisting of \( \phi \) and all subsets of \( X \) not containing a point \( p \) of \( X \).

EXAMPLE 3.5. The excluded point topology is \( L - T_1 \) and \( L - T_2 \) (also is an example of \( Q - L - T_1 \) but not \( T_1 \)).

PROOF. If \( X \) is an infinite set and \( p \) is the excluded point and \( A \subset X \), then:

(i) If \( p \notin A \), we have \( \tau[A] = \tau \cup \{X - B : B \subset A\} \). Thus \( \tau[A] \) is \( T_1 \) but not \( T_2 \).
(ii) If \( p \in A \), then \( A \) is closed, and there are two cases

(a) If \( B \subset A \), \( p \in B \) in this case any open set in \( \tau[A] \) is open in \( \tau \), i.e., \( \tau = \tau[A] \)

(b) If \( B \subset A \), \( p \notin B \) as (i). Thus \( \tau[A] = \tau \cup \{ X - B : B \subset A \} \)

**EXAMPLE 3.6.** Let \( X = [0,1] \) and \( \tau = \{ \phi, X, A \subset X : X - A \text{ is finite} \} \) If we take \( S = (0,1] \), then \( \tau[S] \) is the Discrete space. This example is \( Q - L - T_2 \) but not \( T_2 \)

**THEOREM 3.1.** \( (X, \tau) \) is cid space iff \( \tau = \tau[A] \) whenever \( A \) is a countable infinite subset of \( X \)

**PROOF.** We assume that \( (X, \tau) \) is cid, then \( A \) is closed and discrete subspace. By Theorem 19 we have that \( \tau = \tau[A] \). Conversely we assume that \( \tau = \tau[A] \) By Theorem 18, we have that \( (A, \tau \cap A) \) is a discrete subspace of \( X \) and \( (X, \tau) \) is cid space

**THEOREM 3.2.** Every space \( (X, \tau) \) is \( L-T_0 \)

**PROOF.** Assume that \( x_0 \in X \). We aim to prove that \( \tau[X - \{x_0\}] \) is \( T_0 \). For this purpose let \( x, y \in X, x \neq y \), if \( U \in \tau \) is an open set containing \( x \), then \( U - \{y\} \) is an open set in \( \tau[X - \{x_0\}] \) and not containing \( y \). If \( x_0 = x \), then \( X - \{y\} \) is an open in \( \tau[X - \{x_0\}] \) and not containing \( y \). This completes the proof

The following example illustrates a \( Q - L - T_2 \) space but not \( T_2 \)

**EXAMPLE 3.7.** (Countable complement topology [16]) If \( X \) is an uncountable set, we define the topology of countable complements on \( X \) by declaring open all sets whose complements are countable, together with \( \phi \) and \( X \). \( (X, \tau) \) is \( T_1 \) but not \( T_2 \). Let \( A \subset X \) such that \( X - A \) is countable. For \( x_0 \in X - A \), \( A \cup \{x_0\} \) is \( \tau \)-open, and so \( (A \cup \{x_0\}) - A = \{x_0\} \in \tau[A] \). For \( x_0 \in A \), \( A \) is \( \tau \)-open, which means that \( A - (A - \{x_0\}) = \{x_0\} \) is \( \tau[A] \)-open. Thus \( \tau[A] \) is discrete and consequently \( T_2 \)

**UNSOLVED PROBLEM.** If \( (X, \tau) \) is a space which does not have a property \( P \), what are the properties of the subset \( A \) that make \( (X, \tau[A]) \) have \( P \) (for \( P \) fixed property)

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