ON EXTENSION OF PAIRWISE θ -CONTINUOUS MAPS

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ABSTRACT. The aim of the paper is to find suitable conditions so as to ultimately establish the existence and uniqueness of the extension of a pairwise θ -continuous map onto an arbitrary extension-space of a bitopological space.

KEY WORDS AND PHRASES. Bitopological extension, pairwise θ -continuity, pairwise θ -proper, pairwise $\ast$ -free.

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1. INTRODUCTION.

The problem of extending continuous maps on a topological space X to a given extension X^* of X has been dealt with extensively by many mathematicians. For example it is known (see [2]) that a continuous map $f : X \rightarrow Y$ with Y a compact Hausdorff space can be extended by a continuous map onto an extension X^* of X iff for each pair of closed disjoint subsets A and B of Y , the closures of $f^{-1}(A)$ and $f^{-1}(B)$ in X^* are disjoint. One of the different interesting generalizations of this result arises when continuity and compactness are replaced by θ -continuity and quasi- H -closedness (QHC) respectively under a suitably changed condition (see Rudolf [5]).

Bitopological versions of QHC spaces and θ -continuous functions have been introduced by Mukherjee [4], and Bose and Sinha [1] respectively. It is our purpose here to further generalize the above extension theorem by Rudolf [5]. For this we suitably modify and redefine the appliances used by Rudolf to ultimately establish the existence and uniqueness of the extension of a pairwise θ -continuous map onto an arbitrary extension of a bitopological space under certain conditions.

By spaces X and Y we shall mean bitopological spaces (see Kelly [3]) (X, Q_1, Q_2) and (Y, P_1, P_2) respectively. For any $A \subset X$, Q_i -int A and Q_i -cl A will respectively stand for the interior and closure of A in (X, Q_i) , where $i=1,2$. A set A is called an ij -regularly open set (Singal and Arya [6]) if $A = Q_i$ -int Q_j -cl A , and complement of such a set is called ij -regularly closed where (and also in future discussion) $i, j=1,2$ and $i \neq j$. A space X is called pairwise Hausdorff (Kelly [3]) if for $x, y \in X$ with $x \neq y$, there exist $U \in Q_1$ and $V \in Q_2$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

DEFINITION 1. (see Bose and Sinha [1]) A function (or map) $f : (X, Q_1, Q_2) \rightarrow (Y, P_1, P_2)$ is called ij - θ -continuous if for each $x \in X$ and each P_1 -open neighbourhood (henceforth nbd., for short) U of $f(x)$, there is a Q_1 -open nbd V of x with $f(Q_j\text{-cl}V) \subset P_j\text{-cl}U$. f is called pairwise θ -continuous if it is $l2$ - as well as $2l$ - θ -continuous.

DEFINITION 2. (see Singal and Arya [6]) A subset A of a space (X, Q_1, Q_2) is said to be pairwise dense if every non-empty subset of X which is the intersection of a Q_1 -open set and a Q_2 -open set, has non-void intersection with A .

2. MAIN THEOREM AND ASSOCIATED RESULTS.

DEFINITION 3. A space (X^*, Q_1^*, Q_2^*) is said to be an extension of a space (X, Q_1, Q_2) if $Q_1^*/X = Q_1, Q_2^*/X = Q_2$ and X is pairwise dense in X^* .

For an extension (X^*, Q_1^*, Q_2^*) of (X, Q_1, Q_2) , a map $f : (X, Q_1, Q_2) \rightarrow (Y, P_1, P_2)$ and a point x of X^* (of Y) let N_x^{*i} (resp. N_x^i) the family of all Q_i^* -open (P_i -open) nbds of x in X^* (resp. in Y), for $i=1,2$. For $x \in X^*$, $N^i(f, N_x^{*i})$, shall denote the P_i -open filter on Y generated by the family $\{f(U_x \cap X) : U_x \in N_x^{*i}\}$ ($i=1,2$).

DEFINITION 4. A map $f : X \rightarrow Y$ is ij - θ -proper if for each $x \in X^* - X$, $N^j(f, N_x^{*j})$ has non-void P_i -adherence, where (X^*, Q_1^*, Q_2^*) is an extension of (X, Q_1, Q_2) . The map f is called pairwise θ -proper if for each $x \in X^* - X$, $[N^j(f, N_x^{*j})] \cap [P_j\text{-cl}U : U \in N^2(f, N_x^{*2})] \cap [P_2\text{-cl}U : U \in N^1(f, N_x^{*1})] \neq \emptyset$.

THEOREM 1. Let (X^*, Q_1^*, Q_2^*) be an extension of a space (X, Q_1, Q_2) and $f^* : (X^*, Q_1^*, Q_2^*) \rightarrow (Y, P_1, P_2)$ be an ij - θ -continuous extension of an ij - θ -continuous map $f : X \rightarrow Y$ on X^* . Then $f^*(x \in N^j(f, N_x^{*j}) : U_x \in N_x^{*j})$, for each $x \in X^* - X$.

PROOF. Let $y = f^*(x) \notin P_i\text{-cl}U$, for some $U \in N^j(f, N_x^{*j})$. We consider the P_i -open nbd $U_y = Y - P_i\text{-cl}U$. By ij - θ -continuity of f^* , there exists a Q_i^* -open nbd U_x of x such that $f^*(Q_j^*\text{-cl}U_x) \subset P_j\text{-cl}U_y \subset Y - U$ (since $U \in P_j$), and hence $f^*(U_x \cap X) = f(U_x \cap X) \subset Y - U$. Since $U \in N^j(f, N_x^{*j})$, U contains a set of the form $f(U_x' \cap X)$, for some $U_x' \in N_x^{*j}$. Now, $f(U_x \cap X) \cap f(U_x' \cap X) \subset (Y - U) \cap U = \emptyset$ implies that $f(U_x \cap U_x' \cap X) = \emptyset$, a contradiction because X is pairwise dense in X^* .

LEMMA 1. Let (X^*, Q_1^*, Q_2^*) be an extension of a space (X, Q_1, Q_2) and let $f : (X, Q_1, Q_2) \rightarrow (Y, P_1, P_2)$ be an arbitrary map. Then for each $x \in X^*$ and each $y \in Y$, $y \in N^j(f, N_x^{*j}) : U_y \in N_y^j$ iff $x \in N^j(f, N_x^{*j}) : U_x \in N_x^j$.

PROOF. Let $y \in N^j(f, N_x^{*j}) : U_y \in N_y^j$. Then for each P_i -open nbd U_y of y and each $U \in N^j(f, N_x^{*j})$ we have $U_y \cap U \neq \emptyset$, i.e., $P_j\text{-cl}U_y \cap U \neq \emptyset$ which gives $P_j\text{-cl}U_y \cap f(U_x \cap X) \neq \emptyset$ for each $U_y \in N_y^j$ and each $U_x \in N_x^{*j}$. For otherwise, $P_j\text{-cl}U_y \cap f(U_x \cap X) = \emptyset$ for some $U_y \in N_y^j$ and some $U_x \in N_x^{*j}$. Then $V = Y - P_j\text{-cl}U_y \supset f(U_x \cap X)$ and hence $V \in N^j(f, N_x^{*j})$ for which $V \cap U_y = \emptyset$, a contradiction. Now, $f^{-1}(P_j\text{-cl}U_y) \cap (U_x \cap X) \neq \emptyset$ and hence $f^{-1}(P_j\text{-cl}U_y) \cap U_x \neq \emptyset$, for each $U_x \in N_x^{*j}$ and each $U_y \in N_y^j$. Thus $x \in N^j(f, N_x^{*j}) : U_x \in N_x^j$. Reversing the argument we get the reverse implication.

DEFINITION 5. Let (X^*, Q_1^*, Q_2^*) be an extension of a space (X, Q_1, Q_2) . A map $f : X \rightarrow Y$ is called ij -*-free if for each $x \in X^* - X$, each $y \in Y$ and each ij -regularly closed set A in Y with $y \notin A$, there exists a P_i -open nbd U_y of y with $x \notin Q_j^*\text{-cl} f^{-1}(P_j\text{-cl}U_y) \cap Q_i^*\text{-cl} f^{-1}(A)$. The map f is called pairwise *-free if it is $l2$ - as well as $2l$ -*-free. The map f is called ij -*-proper if f is ij - θ -proper and ij -*-free; f will be called pairwise *-proper if it is pairwise θ -proper and pairwise *-free.

THEOREM 2. Let (X^*, Q_1^*, Q_2^*) be an extension of (X, Q_1, Q_2) . Then for each pairwise

*-proper map f from X to a pairwise Hausdorff space Y , the set $\{ \{ \{ P_1\text{-cl}U : U \in N^2(f, N_X^{*2}) \} \} \cap \{ \{ P_2\text{-cl}V : V \in N^1(f, N_X^{*1}) \} \} \}$ is a singleton for each $x \in X^*-X$.

PROOF. Let $x \in X^*-X$. Since f is pairwise θ -proper, suppose that $y \in \{ \{ \{ P_1\text{-cl}U : U \in N^2(f, N_X^{*2}) \} \} \cap \{ \{ P_2\text{-cl}V : V \in N^1(f, N_X^{*1}) \} \} \}$. By Lemma 1, $x \in \{ \{ \{ Q_2^*\text{-cl} f^{-1}(P_2\text{-cl}U_Y) : U_Y \in N_Y^1 \} \} \cap \{ \{ Q_1^*\text{-cl} f^{-1}(P_1\text{-cl}U_Y) : U_Y \in N_Y^2 \} \} \}$. We consider a point $y' \in Y$ such that $y' \neq y$. Since Y is pairwise Hausdorff, there exists a P_2 -open nbd $V_{y'}$ of y' such that $y \notin P_j\text{-cl}V_{y'}$. Now, f being 12 -*-free there exists a P_j -open nbd U_y of y such that $x \notin Q_2^*\text{-cl} f^{-1}(P_2\text{-cl}U_y) \cap Q_1^*\text{-cl} f^{-1}(P_1\text{-cl}V_{y'})$. Since $x \in \{ \{ \{ Q_2^*\text{-cl} f^{-1}(P_2\text{-cl}U_Y) : U_Y \in N_Y^1 \} \} \}$, $x \notin Q_1^*\text{-cl} f^{-1}(P_1\text{-cl}V_{y'})$ and thus $y' \notin \{ \{ \{ P_2\text{-cl}V : V \in N^1(f, N_X^{*1}) \} \} \}$ (by Lemma 1).

LEMMA 2. For an ij - θ -continuous map $f: X \rightarrow Y$ and $U \in P_j$, $f(Q_i\text{-cl} f^{-1}(U)) \subset P_i\text{-cl}U$.

We are now in a position to prove the main theorem of this paper as follows :

THEOREM 3. Let (X^*, Q_1^*, Q_2^*) be an extension of (X, Q_1, Q_2) . Then each pairwise θ -continuous, pairwise *-proper function $f: X \rightarrow Y$ possesses a pairwise θ -continuous extension $f^*: X^* \rightarrow Y$. The extension is unique if Y is pairwise Hausdorff.

PROOF. For $x \in X$ we take $f^*(x) = f(x)$, and for each $x \in X^*-X$ we choose and fix a point of $\{ \{ \{ P_1\text{-cl}U : U \in N^2(f, N_X^{*2}) \} \} \cap \{ \{ P_2\text{-cl}V : V \in N^1(f, N_X^{*1}) \} \} \}$ and define it to be $f^*(x)$; the latter choice is possible since f is pairwise θ -proper.

We first prove that for each P_j -open set U of Y ,

$$f^*((X^*-X) \cap Q_i^*\text{-cl} f^{-1}(P_i\text{-cl}U)) \subset P_i\text{-cl}U \quad (2.1)$$

If not, then for some $x \in X^*-X$, there exist a P_j -open set U in Y with $x \in (X^*-X) \cap Q_i^*\text{-cl} f^{-1}(P_i\text{-cl}U)$ but $f^*(x) (= y, \text{ say}) \notin P_i\text{-cl}U$. Since f is ij -*-free, there exists a P_i -open nbd V of y such that $x \notin Q_j^*\text{-cl} f^{-1}(P_j\text{-cl}V) \cap Q_i^*\text{-cl} f^{-1}(P_i\text{-cl}U)$. Now since $y = f^*(x) \in \{ \{ \{ P_i\text{-cl}U : U \in N^j(f, N_X^{*j}) \} \} \}$, Lemma 1 gives $x \in Q_j^*\text{-cl} f^{-1}(P_j\text{-cl}V)$ which implies $x \notin Q_i^*\text{-cl} f^{-1}(P_i\text{-cl}U)$, contradicting the choice of x . This proves (2.1).

Now to prove the pairwise θ -continuity of f^* , we first consider $x \in X$. Suppose $f^*(x) (= f(x)) = y$ and let U_y be an arbitrary P_j -open nbd of y . By ji - θ -continuity of f there exists a Q_j -open nbd U_x of x such that $f(Q_i\text{-cl}U_x) \subset P_i\text{-cl}U_y$, i.e.,

$$Q_i\text{-cl}U_x \subset f^{-1}(P_i\text{-cl}U_y) \quad (2.2)$$

Define $U_x^* = \{ U \in Q_j^* : U \cap X = U_x \}$ which is a Q_j^* -open nbd of x . Then using the pairwise denseness of X we have

$$Q_i^*\text{-cl}U_x^* = Q_i\text{-cl}(U_x^* \cap X) = Q_i\text{-cl}U_x \subset Q_i^*\text{-cl}f^{-1}(P_i\text{-cl}U_y) \quad (2.3)$$

Again, $f^*(Q_i^*\text{-cl}U_x^*) = f^*((X^*-X) \cap Q_i^*\text{-cl}U_x^*) \cup f^*(X \cap Q_i^*\text{-cl}U_x^*) = f^*((X^*-X) \cap Q_i^*\text{-cl}U_x^*) \cup f(Q_i\text{-cl}U_x) \subset P_i\text{-cl}U_y$ (by virtue of (2.1), (2.2) and (2.3)).

Next we consider $x \in X^*-X$, and let U_y be a ji -regularly open set containing $f^*(x)$ ($= y$, say). Hence $y \notin Y - U_y$, where $Y - U_y$ is a ji -regularly closed set. Since $y \in \{ \{ \{ P_j\text{-cl}V : V \in N^i(f, N_X^{*i}) \} \} \}$, Lemma 1 gives $x \in \{ \{ \{ Q_i^*\text{-cl} f^{-1}(P_i\text{-cl}U) : U \in N_Y^j \} \} \}$. Then by ji -freeness of f , $x \notin Q_j^*\text{-cl} f^{-1}(Y - U_y)$. Then $U_x = X^* - Q_j^*\text{-cl} f^{-1}(Y - U_y)$ is a Q_j^* -open nbd of x . But $Y = P_i\text{-cl}U_y \cup (Y - U_y)$. Thus $X = f^{-1}(P_i\text{-cl}U_y) \cup f^{-1}(Y - U_y)$. Then $X^* = Q_i^*\text{-cl} f^{-1}(P_i\text{-cl}U_y) \cup Q_j^*\text{-cl} f^{-1}(Y - U_y)$. If not, let $x_0 \in X^*-X$ but $x_0 \notin R.H.S.$ Let V be any Q_i^* -open nbd of x_0 . Since $x_0 \in V \cap [X^* - Q_j^*\text{-cl} f^{-1}(Y - U_y)]$ (note that $x_0 \notin R.H.S.$) and X is pairwise dense in X^* , $V \cap [X^* - Q_j^*\text{-cl} f^{-1}(Y - U_y)] \cap X \neq \emptyset$ which gives $V \cap f^{-1}(P_i\text{-cl}U_y) \neq \emptyset$. Hence $x_0 \in Q_i^*\text{-cl} f^{-1}(P_i\text{-cl}U_y)$, a contradiction. Now since $U_x \cap Q_j^*\text{-cl} f^{-1}(Y - U_y) = \emptyset$, $U_x \subset Q_i^*\text{-cl} f^{-1}(P_i\text{-cl}U_y)$, i.e.,

$$Q_i^*\text{-cl}U_x \subset Q_i^*\text{-cl} f^{-1}(P_i\text{-cl}U_y) \quad (2.4)$$

Again, $U_x \cap X = X - Q_j^* - \text{cl } f^{-1}(Y - U_y) \subset X - f^{-1}(Y - U_y) = f^{-1}(U_y)$. Thus

$$Q_i - \text{cl}(U_x \cap X) \subset Q_i - \text{cl } f^{-1}(U_y) \quad (2.5)$$

Now, $f^*(Q_i^* - \text{cl } U_x) = f^*((X^* - X) \cap Q_i^* - \text{cl } U_x) \cup f^*(X \cap Q_i^* - \text{cl } U_x) = f^*((X^* - X) \cap Q_i^* - \text{cl } U_x) \cup f(Q_i - \text{cl}(U_x \cap X))$ (as X is pairwise dense in X^*) $\subset P_i - \text{cl } U_y$ (by (2.1), (2.4), (2.5) and Lemma 2, noting that f is ij - θ -continuous). If U'_y be any P_j -open nbd of $f^*(x)$, then $U_y = P_j - \text{int } P_i - \text{cl } U'_y$ is a ji -regularly open set containing y . Thus by what we have obtained so far, there is a Q_j^* -open nbd U_x of x with $f^*(Q_i^* - \text{cl } U_x) \subset P_i - \text{cl } U_y = P_i - \text{cl } U'_y$. Hence f^* is ji - θ -continuous at each point of $X^* - X$. Thus we infer that $f^* : X^* \rightarrow Y$ is ji - θ -continuous. The ij - θ -continuity of f^* can similarly be dealt with. The uniqueness of the extension f^* of f follows from Theorems 1 and 2.

REMARK 1. Putting $Q_1 = Q_2$ and $P_1 = P_2$ in the above theorem, we get Theorem 3.1 of Rudolf [5]. If X and Y are topological spaces, then the θ -properness of a map $f : X \rightarrow Y$ is ensured by the QHC property of Y (see [5] for details). In bitopological setting, the definition of pairwise QHC property of Y (cf. [4]) implies that $\bigcap \{P_i - \text{cl } U : U \in N^j(f, N_x^{*j})\} \neq \emptyset$, for $i, j = 1, 2$ ($i \neq j$). But it is not necessary that $\bigcap \{P_1 - \text{cl } U : U \in N^2(f, N_x^{*2})\} \cap \bigcap \{P_2 - \text{cl } U : U \in N^1(f, N_x^{*1})\} \neq \emptyset$. Hence in our case, the role of pairwise θ -properness of f in Theorem 3 cannot be replaced, in general, by pairwise quasi H -closedness of (Y, P_1, P_2) . Nevertheless, taking $Q_1 = Q_2$ and $P_1 = P_2$ we see that every $*$ -free θ -continuous map from a topological space X to any H -closed topological space Y can be extended uniquely over any extension space X^* of X .

EXAMPLE 1. Let $X^* = Y = R$ (=the set of reals), $Q_1^* = P_1$ = the usual topology on R and $Q_2^* = P_2$ = the lower limit topology on R . If X = the set of rationals and $Q_i = Q_i^* / X$, for $i = 1$ and 2 , then clearly (X^*, Q_1^*, Q_2^*) is an extension of (X, Q_1, Q_2) and also, the map $f : (X, Q_1, Q_2) \rightarrow (Y, P_1, P_2)$, defined by $f(x) = x$ ($x \in X$), is pairwise θ -continuous and pairwise $*$ -proper. Since (Y, P_1, P_2) is pairwise Hausdorff, f has a unique pairwise θ -continuous extension over X^* , by Theorem 3.

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