A FIXED POINT THEOREM FOR GENERALIZED METRIC SPACES

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ABSTRACT. In this paper we prove two fixed point theorems for the generalized metric spaces introduced by Dhage.

In a recent paper, Dhage [1] defined a generalized metric space as follows: Let $D : X \times X \times X \rightarrow \mathbb{R}$ with the following properties:

- (i) $D(x, y, z) \geq 0$ for each $x, y, z \in X$, with equality if and only if $x = y = z$,
- (ii) $D(x, y, z) = D(y, x, z) = D(x, z, y) = \dots$ (symmetry)
- (iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$, for each $x, y, z \in X$.

2-metric spaces are defined by a function $d : X \times X \times X \rightarrow \mathbb{R}$ with properties (ii) and (iii) above, and (i) replaced by

- (i') For each distinct pair $x, y \in X$, there exists a $z \in X$ such that $d(x, y, z) \neq 0$, and $d(x, y, z) = 0$ if any two of the triplet x, y, z are equal.

A number of fixed point theorems have been proved for 2-metric spaces. However, Hsiao [2] showed that all such theorems are trivial in the sense that the iterations of f are all colinear. The situation for D -metric spaces is quite different. Some specific examples of D -metric spaces appear in [1].

The purpose of this paper to prove two general fixed point theorems for D -metric spaces.

THEOREM 1. Let X be a complete and bounded D -metric space, f a selfmap of X satisfying

$$D(Tx, Ty, Tz) \leq q \max\{D(x, y, z), D(x, Tx, z), D(y, Ty, z), \\ D(x, Ty, z), D(y, Tx, z)\} \quad (1)$$

for all $x, y, z \in X$, $0 \leq q < 1$. Then T has a unique fixed point p in X , and T is continuous at p .

PROOF. Let $x_0 \in X$ and define $x_{n+1} = Tx_n$. If $x_{n+1} = x_n$ for some n , then T has a fixed point. Assume that $x_{n+1} \neq x_n$ for each n . In (1), setting $x = x_{n-1}$, $y = x_n$, $z = x_{n+p}$, we have

$$D(x_n, x_{n+1}, x_{n+p}) \leq q \max\{D(x_{n-1}, x_n, x_{n+p-1}), D(x_{n-1}, x_n, x_{n+p-1}), \\ D(x_n, x_{n+1}, x_{n+p-1}), D(x_{n-1}, x_{n+1}, x_{n+p-1}), \\ D(x_n, x_n, x_{n+p-1})\}. \quad (2)$$

But

$$D(x_{n-1}, x_n, x_{n+p-1}) \leq q \max\{D(x_{n-2}, x_{n-1}, x_{n+p-2}), D(x_{n-2}, x_{n-1}, x_{n+p-2}), \\ D(x_{n-1}, x_n, x_{n+p-2}), D(x_{n-2}, x_n, x_{n+p-2}), \\ D(x_{n-1}, x_{n-1}, x_{n+p-2})\}, \quad (3)$$

$$\begin{aligned}
D(x_n, x_{n+1}, x_{n+p-1}) &\leq q \max\{D(x_{n-1}, x_n, x_{n+p-2}), D(x_{n-1}, x_n, x_{n+p-2}), \\
&\quad D(x_n, x_{n+1}, x_{n+p-2}), D(x_{n-1}, x_{n+1}, x_{n+p-2}), \\
&\quad D(x_n, x_n, x_{n+p-2})\},
\end{aligned} \tag{4}$$

$$\begin{aligned}
D(x_{n-1}, x_{n+1}, x_{n+p-1}) &\leq q \max\{D(x_{n-2}, x_n, x_{n+p-2}), D(x_{n-2}, x_{n-1}, x_{n+p-2}), \\
&\quad D(x_n, x_{n+1}, x_{n+p-2}), D(x_{n-2}, x_{n+1}, x_{n+p-2}), \\
&\quad D(x_n, x_{n-1}, x_{n+p-1})\},
\end{aligned} \tag{5}$$

and

$$D(x_n, x_n, x_{n+p-1}) \leq q \max\{D(x_{n-1}, x_{n-1}, x_{n+p-2}), D(x_{n-1}, x_n, x_{n+p-2})\}. \tag{6}$$

Substituting (3) - (6) into (2) gives

$$D(x_n, x_{n+1}, x_{n+p}) \leq q^2 \max_{a,b,c} D(x_a, x_b, x_c),$$

where $n-2 \leq a \leq n$, $n-1 \leq b \leq n+1$, and $c = n+p-2$. Continuing this process it follows that

$$D(x_n, x_{n+1}, x_{n+p-1}) \leq q^n \max_{a,b,c} D(x_a, x_b, x_c), \tag{7}$$

where now $0 \leq a \leq n$, $1 \leq b \leq n+1$, and $c = p$. Let $M := \sup_{x,y,z \in X} D(x, y, z)$. Then, it follows from (7) that

$$D(x_n, x_{n+1}, x_{n+p}) \leq q^n M. \tag{8}$$

Using (iii) and (8),

$$\begin{aligned}
D(x_n, x_{n+p}, x_{n+p+t}) &\leq D(x_n, x_{n+p}, x_{n+1}) + D(x_n, x_{n+1}, x_{n+p+t}) + D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\
&\leq 2Mq^n + D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\
&\leq 2Mq^n + D(x_{n+1}, x_{n+p}, x_{n+2}) + D(x_{n+1}, x_{n+2}, x_{n+p+t}) \\
&\quad + D(x_{n+2}, x_{n+p}, x_{n+p+t}) \\
&\leq 2M(q^n + q^{n+1}) + D(x_{n+2}, x_{n+p}, x_{n+p+1}) \leq \dots \\
&\leq 2M(q^n + q^{n+1} + \dots + q^{n+p-1}) + D(x_{n+p-1}, x_{n+p}, x_{n+p+t}) \\
&\leq 2M \sum_{k=n}^{n+p} q^k \leq \frac{2Mq^n}{1-q} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Therefore $\{x_n\}$ is D -Cauchy. Since X is complete, $\{x_n\}$ converges. Call the limit p .

From (1),

$$D(x_n, x_{n+1}, Tp) \leq q \max\{D(x_{n-1}, x_n, p), D(x_n, x_{n+1}, p), D(x_{n-1}, x_{n+1}, p), D(x_n, x_n, p)\}.$$

Taking the limit as $n \rightarrow \infty$, and using the fact that D is continuous, yields $D(p, p, Tp) \leq 0$, which implies that $p = Tp$.

To prove uniqueness, assume that $w \neq p$ is also a fixed point of T . From (1),

$$\begin{aligned}
D(p, w, p) &= D(Tp, Tw, Tp) \\
&\leq q \max\{D(p, w, p), D(p, Tp, p), D(w, Tw, p), D(p, Tw, p), D(w, Tp, p)\} \\
&= q \max\{D(p, w, p), D(w, w, p)\} = qD(w, w, p).
\end{aligned} \tag{9}$$

But

$$\begin{aligned}
D(w, w, p) &= D(w, p, w) = D(Tw, Tp, Tw) \\
&\leq q \max\{D(w, p, w), D(w, Tw, w), D(p, Tp, w), D(w, Tp, w), D(p, Tw, w)\} \\
&= q \max\{D(w, p, w), D(p, p, w)\} = qD(p, p, w)
\end{aligned} \tag{10}$$

Combining (9) and (10) yields $D(p, w, p) \leq q^2 D(p, w, p)$, a contradiction. Therefore $p = w$.

To show that T is continuous at p , let $\{y_n\} \subseteq X$ with $\lim y_n = p$. Then, substituting in (1), with $x = z = p$, $y = y_n$, we obtain

$$D(Tp, Ty_n, Tp) \leq q \max\{D(p, y_n, p), D(p, Tp, p), D(y_n, Ty_n, p), \\ D(p, Ty_n, p), D(y_n, Tp, p)\} \quad (11)$$

Taking the limsup of (11), we obtain

$$\limsup D(p, Ty_n, p) \leq q \max\{0, 0, \limsup D(p, Ty_n, p), 0\},$$

which implies that $\lim Ty_n = p = Tp$, and T is continuous at p .

COROLLARY 1. Let X be a complete and bounded D -metric space, m a positive integer, T a selfmap of X satisfying

$$D(T^m x, T^m y, T^m z) \leq q \max\{D(x, y, z), D(x, T^m x, z), D(y, T^m y, z), \\ D(x, T^m y, z), D(y, T^m x, z)\} \quad (1?)$$

for all $x, y, z \in X$, $0 \leq q < 1$. Then T has a unique fixed point p in X , and T^m is continuous at p .

PROOF. From Theorem 1, T^m has a unique fixed point p , and T^m is continuous at p . But $Tp = T(T^m p) = T^m(Tp)$, and Tp is also a fixed point of T^m . Since the fixed point is unique, $p = Tp$.

THEOREM 2. Let X be a compact D -metric space, f a continuous selfmap of X satisfying

$$D(Tx, Ty, Tz) < \max\{D(x, y, z), D(x, Tx, z), D(y, Ty, z), \\ D(x, Ty, z), D(y, Tx, z)\} \quad (12)$$

for all $x, y, z \in X$. Then T has a unique fixed point p in X .

PROOF. Since X is compact, both sides of (12) are bounded.

Case I. Suppose that the right-hand-side of (12) is positive for all x, y, z in X . Define

$$f(x, y, z) := \frac{D(Tx, Ty, Tz)}{\max\{D(x, y, z), D(x, Tx, z), D(y, Ty, z), D(x, Ty, z), D(y, Tx, z)\}}.$$

Since T and D are continuous, so is f . The compactness of X implies that f assumes its maximum at some point (u, v, w) in X . Call the value c . From (12), it follows that $0 < c < 1$. Thus T now satisfies (1) with $q = c$. By Theorem 1, T has a unique fixed point p .

Case II. Suppose there exists a point (x, y, z) such that the right-hand-side of (12) is zero. Then, in particular, $x = Tx = z$, and x is a fixed point of T . Suppose that w is also a fixed point of T . Then, using the same argument as in Theorem 1, it follows that $x = w$, and the fixed point is unique.

COROLLARY 2. Let X be a compact D -metric space, m a positive integer, T a continuous selfmap of X satisfying

$$D(T^m x, T^m y, T^m z) < \max\{D(x, y, z), D(x, T^m x, z), D(y, T^m y, z), \\ D(x, T^m y, z), D(y, T^m x, z)\} \quad (12)$$

for all $x, y, z \in X$. Then T has a unique fixed point p in X .

The proof of Corollary 2 parallels that of Corollary 1.

Theorem 2.1 and 2.2 of Dhage [1] are special cases of Theorems 1 and 2 of this paper.

There are two limitations involving fixed point theorems on D -metric spaces. The first is that the proof of the existence of a fixed point appears to require that X be bounded. The second is that there is apparently no reasonable contractive definition for a pair of maps on a D -metric space.

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