ON THE BASIS OF THE DIRECT PRODUCT OF PATHS AND WHEELS

A. A. AL-RHAYYEL
Department of Mathematics
Yarmouk University
Irbid, JORDAN

(Received February 24, 1994 and in final form March 22, 1995)

ABSTRACT. The basis number, \( b(G) \), of a graph \( G \) is defined to be the least integer \( k \) such that \( G \) has a \( k \)-fold basis for its cycle space. In this paper we determine the basis number of the direct product of paths and wheels. It is proved that \( P_2 \land W_n \) is planar, and \( b(P_m \land W_n) = 3 \), for all \( m \geq 3 \) and \( n \geq 4 \).

KEY WORDS AND PHRASES. Basis number, cycle space, paths, and wheels.

1991 AMS SUBJECT CLASSIFICATION CODE. 05C99.

1. INTRODUCTION.

Throughout this paper, we consider only finite, undirected, simple graphs. Our notations and terminology will be standard except as indicated. For undefined terms, see [3].

Let \( G \) be a graph, and let \( e_1, \ldots, e_q \) be an ordering of its edges. Then any subset \( H \) of edges in \( G \) corresponds to a \((0,1)\)-vector \( (a_1, \ldots, a_q) \) in the usual way, with \( a_i = 1 \) if \( e_i \in H \) and \( a_i = 0 \) if \( e_i \notin H \). These vectors form a \( q \)-dimensional vector space, denoted by \((Z_2)^q\) over the field of two elements \( Z_2 \).

The vectors in \((Z_2)^q\) which correspond to the cycles in \( G \) generate a subspace called the cycle space of \( G \), denoted by \( C(G) \). We shall say, however, that the cycles themselves, rather than the vectors corresponding to the cycles, generate \( C(G) \). It is well known that (see [3], p. 39)

\[
\dim C(G) = \gamma(G) = q - p + k, \tag{1.1}
\]

where \( q \) is the number of edges, \( p \) is the number of vertices, \( k \) is the number of connected components, and \( \gamma(G) \) is the cyclomatic number of \( G \). A basis for \( C(G) \) is called \( k \)-fold, if each edge of \( G \) occurs in at most \( k \) of the cycles in the basis. The basis number of \( G \) (denoted by \( b(G) \)) is the smallest integer \( k \) such that \( C(G) \) has a \( k \)-fold basis. The fold of an edge \( e \) in a basis \( B \) for \( C(G) \) is defined to be the number of cycles in \( B \) containing \( e \), and denoted by \( f_B(e) \).

The direct product [5] (or conjunction [3]) of two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) is the graph denoted by \( G_1 \land G_2 \) with vertex set \( V_1 \times V_2 \), in which \((v_1, u_1) \) is joined to \((v_2, u_2) \) whenever \( v_1v_2 \in E_1 \) and \( u_1u_2 \in E_2 \). It is clear that \( d_{G_1 \land G_2}(v_1, u_1) = d_{G_1}(v_1) d_{G_2}(u_1) \), where \( d_H(v) \) is the degree of vertex \( v \) in the graph \( H \). Thus the number of edges in \( G_1 \land G_2 \) is \( 2|E_1||E_2| \).

Let \( P_n \) denote a path with \( n \)-vertices, and let \( W_n \) denote a wheel with \( n \) vertices.

The first important result about the basis number was given by MacLane in 1937 (see [4]), when he proved that a graph \( G \) is planar if and only if \( b(G) \leq 2 \). In 1981, Schmeichel [6] proved that \( b(K_n) = 3 \) for \( n \geq 5 \), and for \( m, n \geq 5 \), \( b(K_m \land K_n) = 4 \). In 1983 Banks and Schmeichel [2] proved that \( b(Q_n) = 4 \), for \( n \geq 7 \), where \( Q_n \) is the \( n \)-cube. In 1989, Ali [1] proved that \( b(C_m \land P_n) \leq 2 \), and for all \( m, n \geq 3 \), \( b(C_m \land C_n) = 3 \), where \( C_m \) is a cycle with \( m \) edges. Next we restate Theorem 1 of [2]:

...
THEOREM 1.2. For any connected graph \( G \),
\[
\sum_{v \in V(G)} \left[ \frac{b(G) d(v)}{2} \right] \geq (\text{girth } G) \dim(C(G)),
\]
where \( d(v) \) denotes the degree of a vertex \( v \).

The purpose of this paper is to determine the basis number of \( P_m \land W_n \). In fact it is proved that \( b(P_m \land W_n) = 3 \), for all \( m \geq 3 \). It is also proved that \( P_2 \land W_n \) is planar.

2. MAIN RESULTS

In what follows let \( \{1, 2, \ldots, m\} \) be the vertices of \( P_m \) and let \( \{1, 2, \ldots, n\} \) be the vertices of \( W_n \), with the vertex 1 of \( W_n \), of degree \( n-1 \), and all other vertices of degree 3.

LEMMA 2.1. If \( G = P_2 \land W_n \), then \( G \) is connected.

PROOF. This is clear since \( W_n \) has an odd cycle, namely a 3-cycle (see [3], p. 25). QED

COROLLARY 2.2. If \( G = P_2 \land W_n \), then \( \dim(C(G)) = 2n - 3 \).

PROOF. Just apply (1.1) and Lemma 2.1. QED

THEOREM 2.3. If \( G = P_2 \land W_n \), then \( b(G) = 2 \) and hence \( G \) is planar.

PROOF. Consider the following sets of 4-cycles in \( G \):
\[
E_1 = \{(1,1)(2,i+1)(1,i+2)(2,i+3)(1,1) : i = 1, 2, 3, ..., n-3\}
\]
\[
E_2 = \{(2,1)(1,i+1)(2,i+2)(1,i+3)(2,1) : i = 1, 2, 3, ..., n-3\}
\]
\[
E_3 = \{(1,1)(2,n-1)(1,n)(2,2)(1,1)\}
\]
\[
E_4 = \{(1,1)(2,n)(1,2)(2,3)(1,1)\}
\]
\[
E_5 = \{(2,1)(1,n-1)(2,n)(1,2)(2,1)\}
\]

Let \( B = \bigcup_{j=1}^{5} E_j \), then \( |B| = 2n - 3 = \dim(C(G)) \). Next we show that \( B \) is an independent set of cycles in \( C(G) \).

It is clear that \( E_1 \) consists of \( n-3 \) independent cycles, in fact if \( C \) is a cycle in \( E_1 \), then \( C \) contains the edge \( (1,i+2)(2,i+3) \), which is not an edge of any other cycle in \( E_1 \), hence \( C \) cannot be written as a linear combination of the rest of the cycles in \( E_1 \). A similar argument shows that \( E_2 \) consists of \( n-3 \) independent cycles, and clearly each of \( E_3, E_4 \) and \( E_5 \) consists of exactly one cycle; thus the cycles in each \( E_j (j = 1, 2, 3, 4, 5) \) are independent.

Each cycle of \( E_2 \) contains the edge \( (2,1)(2,i+1) \) which is not in \( E_1 \), hence \( E_1 \cup E_2 \) is an independent set of cycles. The cycle \( E_3 \) contains the edge \( (1,n)(2,2) \), which is not in \( E_1 \cup E_2 \), hence \( E_1 \cup E_2 \cup E_3 \) is an independent set of cycles. The cycle \( E_4 \) contains the edge \( (2,n)(1,2) \), which is not in \( E_1 \cup E_2 \cup E_3 \), hence \( E_1 \cup E_2 \cup E_3 \cup E_4 \) is an independent set of cycles. Finally it is clear that the cycle \( E_5 \) cannot be written as a linear combination of the cycles in \( \bigcup_{j=1}^{4} E_j \). Hence \( B = \bigcup_{j=1}^{5} E_j \) is an independent set of cycles in \( G \), and, since \( |B| = \dim(C(G)) \), \( B \) is a basis for \( C(G) \).

Next, we show that \( B \) is a 2-fold basis of \( C(G) \). Notice that if \( e \) is an edge of \( E = E_1 \cup E_3 \cup E_4 \) of the form \( \{(1,1)(2,i+3) : i = 1, \ldots, n-3\} \) then \( f_E(e) = 2 \), and if \( e_i \) is an edge of \( E \), which is not of the given form, then \( f_E(e_i) = 1 \). Moreover, if \( e \) is an edge of \( E_i = E_2 \cup E_5 \) of the form \( \{(2,1)(i,i+1) : i = 1, \ldots, n-2\} \), then \( f_{E_i}(e) \leq 2 \), and if \( e_i \) is an edge of \( E_i \), which is not of the given form then \( f_{E_i}(e_i) = 1 \), now clearly the edges of the above two forms are disjoint, hence \( f_B(e) \leq 2 \) for any \( e \in G \); thus \( b(G) \leq 2 \). Now \( b(G) > 1 \) because each cycle must have at least 3 edges, which is more than the number of edges in \( G \). Thus \( b(G) = 2 \), and hence \( G \) is planar.

REMARK 2.4. If \( G = P_m \land W_n \), then for all \( m \geq 3, \ n \geq 4 \), we have:
\[
\dim(C(G)) = 3m - 4(m + n) + 5.
\]

THEOREM 2.5. If \( G = P_m \land W_n \), then for all \( m \geq 3, \ b(G) \geq 3 \), and hence \( G \) is nonplanar.
PROOF. If \( b(G) \leq 2 \), then by Theorem 1.2, we have

\[
\sum_{v \in V(G)} d(v) \geq \sum_{v \in V(G)} \left[ \frac{b(G) d(v)}{2} \right] \geq (\text{girth } G) \dim (C(G)) ,
\]

where, \( d(v) \) is the degree of the vertex \( v \), hence

\[
2|E(G)| \geq 4|E(G)| - nm + 1 , \quad (\text{girth } G = 4)
\]

i.e.,

\[
0 \geq 2|E(G)| - 4nm + 4 ,
\]

Now if we evaluate and divide the inequality by four we get:

\[
0 \geq mn - 2m - 2n + 3 = (m - 2)(n - 2) - 1 ,
\]

and since \( n \geq 4 \), we have

\[
1 \geq (m - 1)(n - 2) \geq 2(m - 2) .
\]

Hence \( m \leq 2.5 < 3 \), thus we conclude that if \( m \geq 3 \), then \( b(G) \geq 3 \), hence \( G \) is non planar. QED

**THEOREM 2.6.** If \( G = P_m \triangle W_n \), then for all \( m \geq 3 \), \( b(G) = 3 \).

**PROOF.** The plan here is to give an independent set of cycles \( B \) in \( C(G) \), such that \( |B| = \dim C(G) \), and to show that \( B \) is a 3-fold basis for \( C(G) \). To this end consider the following sets of 4-cycles in \( C(G) \) for \( k = 1, \ldots, m - 1 \), let

\[
E_k = \{(k, 1)(k + 1, i + 1)(k, i + 2)(k + 1, i + 3)(k, 1) : i = 1, \ldots, n - 3 \},
\]

\[
E_{k+1} = \{(k + 1, 1)(k, i + 1)(k + 1, i + 2)(k, i + 3)(k + 1, 1) : i = 1, \ldots, n - 3 \},
\]

\[
A_k = \{(k, 1)(k + 1, n - 1)(k, n)(k + 1, 2)(k, 1) \},
\]

\[
A_{k+1} = \{(k, 1)(k + 1, n)(k, 2)(k + 1, 3)(k, 1) \}, \quad \text{and}
\]

\[
A_{k+1} = \{(k + 1, 1)(k, n - 1)(k + 1, n)(k, 2)(k + 1, 1) \} .
\]

And for \( k = 1, \ldots, m - 2 \), let

\[
D_k = \{(k, 1)(k + 1, 1)(k, i + 1)(k + 1, i + 2)(k, i + 1)(k + 1, 1) : i = 1, \ldots, n - 2 \},
\]

and

\[
D_{k+1} = \{(k + 1, 1)(k + 2, n)(k + 1, n - 1)(k, n)(k + 1, 1) \} .
\]

Let

\[
F_k = E_k \cup E_{k+1} \cup A_k \cup A_{k+1} \cup A_{k+1} (k = 1, \ldots, m - 1) .
\]

\[
F = \bigcup_{k=1}^{m-1} F_k , \quad H_k = D_k \cup D_{k+1} (k = 1, \ldots, m - 2) , \quad H = \bigcup_{k=1}^{m-2} H_k , \quad \text{and let } B = F \cup H . \quad \text{Then}
\]

\[
|B| = |F| + |G| = (m - 1)(2n - 3) + (m - 2)(n - 1) = 3m - 4n - 4m + 5 = \dim C(G) .
\]

For each \( k = 1, \ldots, m - 1 \), notice that \( F_k \) is just a copy of the cycle basis of \( P_2 \triangle W_n \) (with \( \{k, k + 1\} \) as vertices of \( P_2 \)), hence the cycles in each \( F_k \) are independent, and since \( F_{k+1} \) is just a copy of the cycle basis of \( b_2 \triangle W_n \) (with \( \{k, k + 1\} \) as vertices of \( P_2 \)), then it follows that \( \text{If } k \neq \ell \in \{1, \ldots, m - 1 \}, \quad \text{then the cycles in } F_k \text{ are edge disjoint from the cycles in } F_\ell, \quad \text{hence } F \text{ is an independent set of cycles.} \)

Consider \( H_k \), for each \( k = 1, \ldots, m - 2 \), it is clear that the cycles in \( H_k \) are edge disjoint, hence \( H_k \) is an independent set of cycles. Moreover, if \( k \neq \ell \in \{1, \ldots, m - 2 \} \), then the cycles in \( H_k \) are edge disjoint from the cycles in \( H_\ell \), hence \( H = \bigcup_{k=1}^{m-2} H_k \) is an independent set of cycles. Now if \( C \) is any 4-cycle in \( H \), then \( C \) belongs to \( H_k \) for some \( k \), and clearly \( C \) consists of two edges in \( F_k \) and two edges in
\[ F_{k+1}, \text{ hence } C \text{ cannot be written as a linear combination of cycles in } F, \text{ hence } B = F \cup H \text{ is an independent set of cycles with } |B| = \dim C(G) \text{ Thus } B \text{ is a basis for } C(G). \]

It remains to show that \( B \) is a 3-fold basis for \( C(G) \), but this is clear since if \( e \) is an edge of \( G \), then it follows from the result when \( m = 2 \) that \( f_F(e) \leq 2 \), and \( f_H(e) \leq 1 \), hence \( f_B(e) \leq 3 \) (i.e., \( b(G) \leq 3 \)). Now combining this with Theorem 2.5, we see that \( B \) is a 3-fold basis for \( C(G) \) QED

**REFERENCES**

Submit your manuscripts at http://www.hindawi.com