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ON THE BASIS OF THE DIRECT PRODUCT OF PATHS AND WHEELS

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ABSTRACT. The basis number, $b(G)$, of a graph G is defined to be the least integer k such that G has a k-fold basis for its cycle space In this paper we determine the basis number of the direct product of paths and wheels. It is proved that $P_2 \wedge W_n$is planar, and $b(P_m \wedge W_n) = 3$, for all $m \geq 3$ and $n \geq 4$.

KEY WORDS AND PHRASES. Basis number, cycle space, paths, and wheels.
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1. INTRODUCTION.

Throughout this paper, we consider only finite, undirected, simple graphs. Our notations and terminology will be standard except as indicated For undefined terms, see [3].

Let G be a graph, and let $e_1, ..., e_q$ be an ordering of its edges. Then any subset H of edges in G corresponds to a (0,1)-vector $(a_1, ..., a_q)$ in the usual way, with $a_i = 1$ if $e_i \in H$ and $a_i = 0$ if $e_i \notin H$. These vectors form a q-dimensional vector space, denoted by $(\mathbf{Z}_2)^q$ over the field of two elements $\mathbf{Z}_2$.

The vectors in $(\mathbf{Z}_2)^q$ which corresponds to the cycles in G generate a subspace called the cycle space of G, denoted by $C(G)$. We shall say, however, that the cycles themselves, rather than the vectors corresponding to the cycles, generate $C(G)$. It is well known that (see [3], p. 39)

$$\dim C(G) = \gamma(G) = q - p + k\,, \tag{1.1}$$

where q is the number of edges, p is the number of vertices, k is the number of connected components, and $\gamma(G)$ is the cyclomatic number of G. A basis for $C(G)$ is called k-fold, if each edge of G occurs in at most k of the cycles in the basis. The basis number of G (denoted by $b(G)$) is the smallest integer k such that $C(G)$ has a k-fold basis. The fold of an edge e in a basis B for $C(G)$ is defined to be the number of cycles in B containing e, and denoted by $f_B(e)$

The direct product [5] (or conjunction [3]) of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph denoted by $G_1 \wedge G_2$ with vertex set $V_1 \times V_2$, in which (v_1, u_1) is joined to (v_2, u_2) whenever $v_1v_2 \in E_1$ and $u_1u_2 \in E_2$. It is clear that $d_{G_1 \wedge G_2}(v_i, u_j) = d_{G_1}(v_i)\, d_{G_2}(u_j)$, where $d_H(v)$ is the degree of vertex v in the graph H. Thus the number of edges in $G_1 \wedge G_2$ is $2|E_1|\,|E_2|$.

Let P_m denote a path with m-vertices, and let W_n denote a wheel with n vertices.

The first important result about the basis number was given by MacLane in 1937 (see [4]), when he proved that a graph G is planar if and only if $b(G) \leq 2$ In 1981, Schmeichel [6] proved that $b(k_n) = 3$ for $n \geq 5$, and for $m, n \geq 5$, $b(k_{m,n}) = 4$. In 1982 Banks and Schmeichel [2] proved that $b(Q_n) = 4$, for $n \geq 7$, where Q_n is the n-cube In 1989, Ali [1] proved that $b(C_m \wedge P_n) \leq 2$, and for all $m, n \geq 3$, $b(C_m \wedge C_n) = 3$, where C_m is a cycle with m edges. Next we restate Theorem 1 of [2]:

THEOREM 1.2. For any connected graph G,

$$\sum_{v \in V(G)} \left[\frac{b(G)\, d(v)}{2} \right] \geq (\operatorname{girth} G) \dim(C(G)) ,$$

where $d(v)$ denotes the degree of a vertex v.

The purpose of this paper is to determine the basis number of $P_m \wedge W_n$. In fact it is proved that $b(P_m \wedge W_n) = 3$, for all $m \geq 3$. It is also proved that $P_2 \wedge W_n$ is planar.

2. MAIN RESULTS

In what follows let $\{1, 2, ..., m\}$ be the vertices of P_m and let $\{1, 2, ..., n\}$ be the vertices of W_n, with the vertex 1 of W_n of degree $n-1$, and all other vertices of degree 3.

LEMMA 2.1. If $G = P_2 \wedge W_n$, then G is connected.

PROOF. This is clear since W_n has an odd cycle, namely a 3-cycle (see [3], p. 25). QED

COROLLARY 2.2. If $G = P_2 \wedge W_n$, then $\dim C(G) = 2n - 3$.

PROOF. Just apply (1.1) and Lemma 2.1. QED

THEOREM 2.3. If $G = P_2 \wedge W_n$, then $b(G) = 2$ and hence G is planar.

PROOF. Consider the following sets of 4-cycles in G:

$$\begin{aligned}
E_1 &= \{(1,1)(2,i+1)(1,i+2)(2,i+3)(1,1) : i = 1,2,3,...,n-3\} \\
E_2 &= \{(2,1)(1,i+1)(2,i+2)(1,i+3)(2,1) : i = 1,2,3,...,n-3\} \\
E_3 &= \{(1,1)(2,n-1)(1,n)(2,2)(1,1)\} \\
E_4 &= \{(1,1)(2,n)(1,2)(2,3)(1,1)\} \\
E_5 &= \{(2,1)(1,n-1)(2,n)(1,2)(2,1)\}
\end{aligned}$$

Let $B = \bigcup_{j=1}^{5} E_j$, then $|B| = 2n - 3 = \dim C(G)$. Next we show that B is an independent set of cycles in $C(G)$.

It is clear that E_1 consists of $n-3$ independent cycles, in fact if C is a cycle in E_1, then C contains the edge $(1, i+2)(2, i+3)$, which is not an edge of any other cycle in E_1, hence C cannot be written as a linear combination of the rest of the cycles in E_1. A similar argument shows that E_2 consists of $n-3$ independent cycles, and clearly each of E_3, E_4 and E_5 consists of exactly one cycle; thus the cycles in each $E_j (j = 1, ..., 5)$ are independent.

Each cycle of E_2 contains the dege $(2,1)(2,i+1)$ which is not in E_1, hence $E_1 \cup E_2$ is an independent set of cycles. The cycle E_3 contains the dege $(1,n)(2,2)$, which is not in $E_1 \cup E_2$, hence $E_1 \cup E_2 \cup E_3$ is an independent set of cycles. The cycle E_4 contains the dege $(2,n)(1,2)$, which is not in $E_1 \cup E_2 \cup E_3$, hence $E_1 \cup E_2 \cup E_3 \cup E_4$ is an independent set of cycles. Finally it is clear that the cycle E_5 cannot be written as a linear combination of the cycles in $\bigcup_{j=1}^{4} E_j$. Hence $B = \bigcup_{j=1}^{5} E_j$ is an independent set of cycles in G, and, since $|B| = \dim C(G)$, B is a basis for $C(G)$.

Next, we show that B is a 2-fold basis of $C(G)$. Notice that if e is an edge of $E = E_1 \cup E_3 \cup E_4$ of the form $\{(1,1)(2,i+3) : i = 1, ..., n-3\}$ then $f_E(e) = 2$, and if e_i is an edge of E, which is not of the given form, then $f_E(e_i) = 1$. Moreover, if e is an edge of $E_i = E_2 \cup E_5$ of the form $\{(2,1)(1,i+1) : i = 1, ..., n-2\}$, then $f_{E_i}(e) \leq 2$, and if e_i is an edge of E_i, which is not of the given form then $f_{E_i}(e_i) = 1$, now clearly the edges of the above two forms are disjoint, hence $f_B(e) \leq 2$ for any $e \in G$; thus $b(G) \leq 2$. Now $b(G) > 1$ because each cycle must have at least 3 edges, which is more than the number of edges in G. Thus $b(G) = 2$, and hence G is planar.

REMARK 2.4. If $G = P_m \wedge W_n$, then for all $m \geq 3$, $n \geq 4$, we have:

$$\dim C(G) = 3m - 4(m+n) + 5.$$

THEOREM 2.5. If $G = P_m \wedge W_n$, then for all $m \geq 3$, $b(G) \geq 3$, and hence G is nonplanar.

PROOF. If $b(G) \leq 2$, then by Theorem 1 2, we have

$$\sum_{v \in V(G)} d(v) \geq \sum_{v \in V(G)} \left[\frac{b(G)\, d(v)}{2}\right] \geq (\operatorname{girth} G) \dim (C(G)) ,$$

where, $d(v)$ is the degree of the vertex v, hence

$$2|E(G)| \geq 4[|E(G)| - nm + 1] , \quad (\operatorname{girth} G = 4)$$

i e ,

$$0 \geq 2|E(G)| - 4nm + 4 ,$$

Now if we evaluate and divide the inequality by four we get:

$$0 \geq mn - 2m - 2n + 3 = (m-2)(n-2) - 1 ,$$

and since $n \geq 4$, we have

$$1 \geq (m-1)(n-2) \geq 2(m-2) .$$

Hence $m \leq 2.5 < 3$, thus we conclude that if $m \geq 3$, then $b(G) \geq 3$, hence G is non planar. QED

THEOREM 2.6. If $G = P_m \wedge W_n$, then for all $m \geq 3$, $b(G) = 3$.

PROOF. The plan here is to give an independent set of cycles B in $C(G)$, such that $|B| = \dim C(G)$, and to show that B is a 3-fold basis for $C(G)$ To this end consider the following sets of 4-cycles in $C(G)$ for $k = 1, \ldots, m-1$, let

$$\begin{aligned}
E_k &= \{(k,1)(k+1,i+1)(k,i+2)(k+1,i+3)(k,1) : i = 1, \ldots, n-3\} , \\
E_{k¿} &= \{(k+1,1)(k,i+1)(k+1,i+2)(k,i+3)(k+1,1) : i = 1, \ldots, n-3\} , \\
A_k &= \{(k,1)(k+1,n-1)(k,n)(k+1,2)(k,1)\} , \\
A_{k¿} &= \{(k,1)(k+1,n)(k,2)(k+1,3)(k,1)\} , \text{ and} \\
A_{k¿¿} &= \{(k+1,1)(k,n-1)(k+1,n)(k,2)(k+1,1)\} .
\end{aligned}$$

And for $k = 1, \ldots, m-2$, let

$$D_k = \{(k+1,1)(k+2,i+1)(k+1,i+2)(k,i+1)(k+1,1) : i = 1, \ldots, n-2\} ,$$

and

$$D_{k¿} = \{(k+1,1)(k+2,n)(k+1,n-1)(k,n)(k+1,1)\} .$$

Let

$$F_k = E_k \cup E_{k¿} \cup A_k \cup A_{k¿} \cup A_{k¿¿} \, (k = 1, \ldots, m-1) .$$

$F = \bigcup_{k=1}^{m-1} F_k$, $H_k = D_k \cup D_{k¿} \, (k = 1, \ldots, m-2)$, $H = \bigcup_{k=1}^{m-2} H_k$, and let $B = F \cup H$. Then

$$|B| = |F| + |G| = (m-1)(2n-3) + (m-2)(n-1) = 3m - 4n - 4m + 5 = \dim C(G) .$$

For each $k = 1, \ldots, m-1$, notice that F_k is just a copy of the cycle basis of $P_2 \wedge W_n$ (with $\{k, k+1\}$ as vertices of P_2), hence the cycles in each F_k are independent, and since F_ℓ is just a copy of the cycle basis of $b_2 \wedge W_n$ (with $\{\ell, \ell+1\}$ as vertices of P_2), then it follows that If $k \neq \ell$ in $\{1, \ldots, m-1\}$, then the cycles in F_k are edge disjoint from the cycles in F_ℓ, hence F is an independent set of cycles.

Consider H_k, for each $k = 1, \ldots, m-2$, it is clear that the cycles in H_k are edge disjoint, hence H_k is an independent set of cycles. Moreover, if $k \neq \ell$ in $\{1, \ldots, m-2\}$, then the cycles in H_k are edge disjoint from the cycles in H_ℓ, hence $H = \bigcup_{k=1}^{m-2} H_k$ is an independent set of cycles Now if C is any 4-cycle in H, then C belongs to H_k for some k, and clearly C consists of two edges in F_k and two edges in

F_{k+1}, hence C cannot be written as a linear combination of cycles in F, hence $B = F \cup H$ is an independent set of cycles with $|B| = \dim C(G)$ Thus B is a basis for $C(G)$.

It remains to show that B is a 3-fold basis for $C(G)$, but this is clear since if e is an edge of G, then it follows from the result when $m = 2$ that $f_F(e) \leq 2$, and $f_H(e) \leq 1$, hence $f_B(e) \leq 3$ (i e., $b(G) \leq 3$). Now combining this with Theorem 2 5, we see that B is a 3-fold basis for $C(G)$ QED

REFERENCES

[1] ALI, A. ALI, The basis numbers of the direct products of paths and cycles, *Ars Combinatoria*, **27** (1989), 155-164.

[2] BANKS, J. A. and SCHMEICHEL, E. F , The basis number of the n-cube, *J. Combin. Theory*, Ser. B **33 2** (1982), 95-100.

[3] HARRAY, F., *Graph Theory*, 2nd ed., Addison-Wesley, Reading, Massachusetts, 1971.

[4] MacLANE, S., A combinatorial condition for planar graphs, *Fund. Math.* **28** (1937), 22-32.

[5] NESETRIL, J. and PULTER, A., Products and other representation of graphs and related characteristics, in *Algebraic Methods in Graph Theory*, Colloquia Mathematica Societais Ja¿nos Bolyai, Szeged Hungary, **25** (1978), 571-598.

[6] SCHMEICHEL, E. F., The basis number of a graph, *J. Combin. Theory*, Ser. B **30 2** (1981), 123-129.