UNIVALENCE FOR CONVOLUTIONS

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ABSTRACT. The radius of univalence is found for the convolution $f*g$ of functions $f \in S$ (normalized univalent functions) and $g \in C$ (close-to-convex functions). A lower bound for the radius of univalence is also determined when f and g range over all of S . Finally, a characterization of C provides an inclusion relationship.

KEY WORDS AND PHRASES. Univalent, convolution.

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1. INTRODUCTION.

Denote by S the family consisting of functions $f(z) = z + \dots$ that are analytic and univalent in $\Delta = \{z: |z| < 1\}$ and by K, S^* , and C the subfamilies of functions that are, respectively, convex, starlike, and close-to-convex in Δ . It is well known that $K \subset S^* \subset C \subset S$. The convolution of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as the power series

$$f(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The Koebe function $k(z) = z/(1-z)^2$ often plays an extremal role in the family S . This enables us to show it to be extreme in many convolution problems. For example, the modulus of the n th coefficient for $f*g$, f and g in S , is n^2 and is attained when $f = g = k$. Similarly, $|f*g|$ takes its maximum and minimum on the circle $|z| = r$ when $f = g = k$.

A question was raised in [4] as to whether

$$\min_{|z|=r} \operatorname{Re} \frac{(f*g)(z)}{z} = \min_{|z|=r} \operatorname{Re} \frac{(f*k)(z)}{z} = \min_{|z|=r} \operatorname{Re} f'$$

when f and g are taken over all of S . The classical rotation theorem for $f \in S$ leads to the sharp result that $\operatorname{Re} f'(z) \geq 0$ when $|z| \leq \sin(\pi/8)$. This was generalized in [4] to $\operatorname{Re} \frac{(f*g)(z)}{z} \geq 0$ for $|z| \leq \sin(\pi/8)$ when $f \in S$ and $g \in S^*$, but could not be extended to $g \in S$ or even to $g \in C$. In particular, functions $f, g \in C$ were found for which $\operatorname{Re} \frac{(f*g)(z)}{z} < 0$ at some point z_0 , $|z_0| < \sin(\pi/8)$.

In this note, we investigate the radius of univalence for $f*g$, f and g in S . For $f \in S$ and $g = k$, the Koebe function, $f*k$ is univalent in the disk $|z| < 2 - \sqrt{3}$. We prove that $g = k$ can be replaced by any $g \in C$, but we cannot settle if this extends to arbitrary $g \in S$. We do show, however, that $f*g$ is univalent for at least $|z| < .8(2 - \sqrt{3})$.

2. MAIN RESULTS.

THEOREM 1. If $f \in S$ and $g \in C$, then $f*g$ is univalent in $|z| < 2 - \sqrt{3}$. The result is sharp.

PROOF. It is well known that f is convex in $|z| < r$ if and only if zf' is starlike in $|z| < r$ and that the radius of convexity of S is $2 - \sqrt{3}$. Thus, $f*k = zf'$ has radius of starlikeness (and hence radius of univalence) at least $2 - \sqrt{3}$, the radius of convexity for $f \in S$. Since

$$(k*k)' = (zk')' = \frac{1+4z+z^2}{(1-z)^4} = 0 \text{ at } z = -(2 - \sqrt{3}),$$

the radius of univalence of $f*g$ for $f, g \in S$ can be no greater than $r = 2 - \sqrt{3}$.

When $f \in S$, we have $f(az)/a \in K$ for $a = 2 - \sqrt{3}$. Hence, by a theorem of Ruscheweyh and Sheil-Small [3], if $f \in S$ and $g \in C$ then

$$\frac{f(az)}{a} * g(z) \in C \subset S.$$

Thus, $f*g$ is univalent for $|z| < 2 - \sqrt{3}$, and the proof is complete.

In our next theorem, we replace C with S in the hypothesis and this leads to a weaker conclusion.

THEOREM 2. Denote by r_0 the largest value for which $f*g$ is univalent in $|z| < r_0$ for all $f, g \in S$. Then $.8(2 - \sqrt{3}) < r_0 \leq 2 - \sqrt{3}$.

PROOF. The upper bound was found in Theorem 1. Krzyz [1] determined the radius of close-to-convexity for S to be $t_0 = 0.80 +$. Since $f(az)/a \in K$, $a = 2 - \sqrt{3}$, and $g(t_0z)/t_0 \in C$, we have from the Ruscheweyh and Sheil-Small theorem [3] that $\frac{f(az)}{a} * \frac{g(t_0z)}{t_0} \in C$, which shows that $f*g$ is univalent for $|z| < t_0(2 - \sqrt{3})$. This furnishes us with the lower bound, and the proof is complete.

Though we are unable to prove that $r_0 = 2 - \sqrt{3}$ in Theorem 2, the lower bound on r_0 most certainly can be improved. Ruscheweyh defined the family M consisting of normalized functions f by

$$M = \{f: f*g \neq 0; g \in S^*, 0 < |z| < 1\}.$$

He proved the proper inclusions $C \subset M \subset S$ and that $f*g \in M$ for $f \in K$ and $g \in M$ [2]. Hence, if t_1 is the largest value for which $g(t_1z)/t_1 \in M$ when $g \in S$, methods identical to those of Theorem 2 show that $f*g$ is univalent in $|z| < t_1(2 - \sqrt{3})$ for $f, g \in S$. Unfortunately the value of t_1 , the radius of " M -ness" for S , is unknown.

3. A CHARACTERIZATION OF C .

The inclusion $C \subset M$ is not obvious and was proved by Ruscheweyh using his duality principle [2]. Our final result is a characterization of C that leads to a more elementary proof that $C \subset M$. We make use of a result found in [3].

LEMMA 3. If $\phi \in K$, $\Psi \in S^*$, and F is analytic with $\operatorname{Re} F > 0$ for $z \in \Delta$, then

$$\operatorname{Re} \frac{\phi * F \Psi}{\phi * \Psi} > 0.$$

THEOREM 3. A function $f \in C$ if and only if to each $g \in S^*$ we may associate an $h \in S^*$ for which $\operatorname{Re} \frac{f * g}{h} > 0$, $z \in \Delta$.

PROOF. To show that the condition is sufficient for f to be in C , we choose $g(z) = z/(1-z)^2 \in S^*$. Then $\operatorname{Re} \frac{f * g}{h} = \operatorname{Re} \frac{zf'}{h} > 0$, which means that $f \in C$.

On the other hand, if $f \in C$ we can find a $\Psi \in S^*$ for which $\operatorname{Re} zf'/\Psi > 0$. Set $F(z) = zf'(z)/\Psi(z)$. Then for $g \in S^*$ there corresponds $\phi \in K$ such that $z\phi' = g$. Note that $f * g = zf' * \phi = \phi * F\Psi$ and that $h = \phi * \Psi \in S^*$. By Lemma A,

$$\operatorname{Re} \frac{\phi * F\Psi}{\phi * \Psi} = \operatorname{Re} \frac{f * g}{h} > 0,$$

and the proof is complete

COROLLARY. $C \subset M$

PROOF. Since $\operatorname{Re} \frac{f * g}{h} > 0 \Rightarrow f * g \neq 0$, the result follows from Theorem 3.

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