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ON COMPLETELY 0-SIMPLE SEMIGROUPS

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ABSTRACT. Let S be a completely 0-simple semigroup and F be an algebraically closed field. Then for each 0-minimal right ideal M of S, $M = B \cup C \cup \{0\}$, where B is a right group and C is a zero semigroup. Also, a matrix representation for S other than Rees matrix is found for the condition that the semigroup ring $R(F, S)$ is semisimple Artinian.

KEY WORDS AND PHRASES. Completely 0-simple semigroups, 0-minimal right ideals, right groups, zero semigroups, representation of semigroups, semisimple Artinian semigroup rings.
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1. INTRODUCTION.

A semigroup S is a set of elements together with an associative, binary operation defined on S. A nonempty subset A of a semigroup S is a left (right) ideal of S if $SA \subseteq A(AS \subseteq A)$. A is a two-sided ideal of S if it is both a left ideal and a right ideal of S. A is said to be a minimal (left, right) ideal of S if, for any (left, right) ideal B, $B \subseteq A$ implies $B = A$. A (left, right) ideal A of S is said to be 0-minimal if whenever there is a (left, right) ideal B of S contained in A, either $B = A$ or $B = \{0\}$. S is a 0-simple semigroup if $S^2 \neq \{0\}$ and $\{0\}$ is the only proper ideal of S.

An element e in S is called an idempotent if $e^2 = e$. Let E be the set of idempotents. Define $e \leq f$ if $ef = e = fe$. Then a nonzero idempotent is said to be primitive if it is minimal with respect to $\leq$ and S is said to be completely 0-simple if it is 0-simple and contains a primitive idempotent.

Let F be a field. A semigroup ring $R(F, S)$ is an associative F-algebra with the semigroup S as its basis and with multiplication defined distributively using the semigroup multiplication in S. If I is a (left, right) ideal of S then the semigroup ring $R(F, I)$ is a (left, right) ideal of $R(F, S)$. For each $\tilde{a}$ in $R(F, S)$, $\tilde{a} = \sum_{x \in S, \alpha_x \in F} \alpha_x x$ such that only a finite number of α_x's are nonzero. The set

$$Supp(\tilde{a}) = \{x \in S | \alpha_x \neq 0, \tilde{a} = \sum_{x \in S, \alpha_x \in F} \alpha_x x\} \tag{1.1}$$

is called the support of $\tilde{a}$ and by the length of $\tilde{a}$ we mean the number of distinct elements in

$Supp(\tilde{a})$ and denote it by $\ell(\tilde{a})$.

An $n \times n$ matrix $A = (a_{ij})$ is called a mono-row matrix if at most one row of A contains nonzero entries; i.e. $a_{ij} = 0$ for all i, j except $i = i_0$ for some i_0. Let T be a semigroup and let $\mathcal{M}(n, T^0)$ be the set of all the $n \times n$ mono-row matrices over T^0. Then $\mathcal{M}(n, T^0)$ is a semigroup with matrix multiplication as its operation.

Throughout this paper, S denotes a completely 0-simple semigroup, F denotes an algebraically closed field, $R(F,T)$ means the semigroup algebra generated by a semigroup T, and $R = R(F, S)$.

2. O-MINIMAL RIGHT IDEALS.

Since S is completely 0-simple, it is shown in [1] that S is regular and contains at least one 0-minimal right ideal. Let M be such a 0-minimal right ideal. Then $M = eS$ for some primitive idempotent e which serves as a left identity in M. Suppose there exists a nonzero element a in S such that $aS = 0$. Then $a \notin aSa = \{0\}$ which contradicts the regularity of S. Therefore for all nonzero a in $S, aS \neq 0$. Hence, $M = B \cup C \cup \{0\}$ where

$$B = \{b \in M | bS = M = bM\} \tag{2.1}$$

and

$$C = \{c \in M | cS = M \text{ and} cM = 0\}. \tag{2.2}$$

PROPOSITION 2.1. B is a right group; i.e. $B \cong G \times E$ where G is a group and E is a right zero semigroup.

PROOF. B is a semigroup because, for all b_1, $b_2 \in B$,

$$(b_1 b_2)S = b_1(b_2 S) = b_1 M = M \tag{2.3}$$

and

$$(b_1 b_2)M = b_1(b_2 M) = b_1 M = M. \tag{2.4}$$

In order to be a right group, B has to be right simple and contain a primitive idempotent. Obviously, the generator e of M is in B for $eS = M = eM$. So $B \neq \emptyset$. Given any $b \in B$, if $bm_1 = bm_2$, for m_1, $m_2 \in M$, then $ebem_1 = ebem_2$, since e is a left identity of M. But from [1] we know that eSe is a group with 0 and identity e. So ebe must have an inverse b' in eSe. Thus

$$m_1 = em_1 = b'(ebe)m_1 = b'(ebe)m_2 = em_2 = m_2. \tag{2.5}$$

Therefore $bm_1 = bm_2$ if and only if $m_1 = m_2$. Now given $a, b \in B$, $a \in M = bM$ implies $a = bm$ for some $m \in M$. m must be in B; otherwise $aM = b(mM) = 0$ contradicts the assumption that $a \in B$. Hence $bB = B$ for all $b \in B$. Therefore, B is a right group and $B \cong G \times E$ where G is a group and E is a right zero semigroup.

Let g_0 be the identity of G. Then (g_0, e), for any $e \in E$, is a left identity of B and of M. Given any $b \in B$ and $c \in C$, $(bc)S = b(cS) = bM = M$ and $(bc)M = b(cM) = 0$ imply that

$C = bC$. In particular, $(g_0, e)c = c$. Conversely, if $(g, e)c = c$ for some $g \in G$, then $cs = (g_0, e)$ for some $s \in S$ because $cS = M$. Hence

$$(g, e) = (g, e)(g_0, e) = (g, e)cs = cs = (g_0, e); \tag{2.6}$$

i.e. $g = g_0$. So $(g, e)c = c$ for any $c \in C \iff g = g_0$. Using this result and denoting $d_g = (g, e)d$ for $g \in G$ and $d \in C$, we get

$$d_g = d_h \iff d = (g^{-1}h, e)d \iff g^{-1}h = g_0 \iff g = h. \tag{2.7}$$

PROPOSITION 2.2. Fix an element $e \in E$. Then there exists a subset D in C such that every $c \in C$ can be uniquely expressed by d_g for some $g \in G$ and $d \in D$.

PROOF. For the fixed e, consider the collection

$$\mathcal{A} = \{A \subseteq C | (g, e)A \subseteq C \text{ and } (g, e)a_1 \neq (h, e)a_2 \text{ for all } g, h \in G \text{ and } a_1 \neq a_2 \in A\}. \tag{2.8}$$

Suppose $\mathcal{B}$ is a chain in $\mathcal{A}$. Then for any distinct a_1 and a_2 in $\cup\mathcal{B}$ there exist A_1, $A_2 \in \mathcal{B}$ such that $a_1 \in A_1$ and $a_2 \in A_2$. Without loss of generosity, assume $A_1 \subseteq A_2$. Then $a_1, a_2 \in A_2$. Then $(g, e)a_1 \neq (h, e)a_2$ for all g, $h \in G$; i.e. $\cup\mathcal{B} \subseteq \mathcal{A}$. By Zorn's Lemma, $\mathcal{A}$ contains a maximal element D and so every $c \in C$ can be uniquely expressed as $c = d_g$ for some $g \in G$ and $d \in D$. Otherwise $D \cup \{c\} \subseteq \mathcal{A}$ which is contradictary to the nature of D.

With the result of Proposition 2.2, let us denote $(g, d) = (g, e)d$ for each $(g, e)d \in C$. Then

$$(h, f)(g, d) = (h, f)(g, e)(g_0, e)d = (hg, d) \tag{2.9}$$

for all $g, h \in G$, $f \in E$, and $d \in D$. We conclude that $(g, f)(h, x) = (gh, x)$ for all $g, h \in G, f \in E$, and $x \in E \cup D$.

According to the Rees Theorem in [1], a completely 0-simple semigroup can be represented by a regular Rees $m \times n$ matrix semigroup, $M^0(H; m, n; P)$ over the group H, with an $n \times m$ sandwich matrix P. While the group H means the $\mathcal{H}$-class of an idempotent e, we can see that $G \times \{e\} = eSe = H$.

For each $s \in S$, sM is either $\{0\}$ or a 0-minimal right ideal. So $S = \cup s_i M$, where $s_1 = (g_0, e)$ and $s_i M \neq s_j M$ for all $i \neq j$. Furthermore, $s_i M = B_i \cup C_i \cup \{0\}$ for each i with

$$B_i = \{b \in s_i M | bS = s_i M = b(s_i M)\} \tag{2.10}$$

and

$$C_i = \{c \in s_i M | cS = s_i M \text{ and } c(s_i M) = \{0\}\}. \tag{2.11}$$

Choose $s \in S$ so that sM is 0-minimal. Note that $s \in sM$; otherwise $s = tm$ for some other 0-minimal right ideal tM. If $m \in B$ then $sM = tmM = tM$; while $m \in C$ implies $sM = tmM = 0$. So $sS = sM = s_i M$ for some i. Given $m \in M$, we have

$$sm \in B_i \iff (sm)S = s_i M = sM = (sm)sM \iff ms \in B, \tag{2.12}$$

$$\text{while } sm = 0 \iff (sm)S = 0 \iff s(mS) = 0 \iff m = 0. \tag{2.13}$$

Now if $ms \in C$ then $M = (ms)S = (ms)M = 0$ causes a contradiction. So when $sm \in C_i$, $ms = 0$. From here, we get

$$sm_1 = sm_2 \iff ((g^{-1}, e)m)sm_1 = ((g^{-1}, e)m)sm_2, \tag{2.14}$$

for some $m \in M$ such that

$$\begin{aligned} ms = (g, e) &\iff (g^{-1}, e)(ms)m_1 = (g^{-1}, e)(ms)m_2 \\ &\iff (g_0, e)m_1 = (g_0, e)m_2 \iff m_1 = m_2. \end{aligned} \tag{2.15}$$

3. SEMISIMPLE ARTINIAN SEMIGROUP ALGEBRAS.

Now consider the semigroup algebra $R = R(F, S)$ where S is a completely 0-simple semigroup. We learned from [2] that a simple ideal in a semisimple Artinian ring is isomorphic to a matrix ring. With this in mind, we would like to see if this matrix ring can help us find a matrix semigroup representing S.

First, let us look at two important properties.

PROPOSITION 3.1. (see [3]) If $R(F, S)$ is right Artinian, then S is finite.

PROPOSITION 3.2. (see [1]) $R(F, G)$ is semisimple Artinian if and only if char$F$ does not divide $|G|$.

When S is finite, $w_x = \sum_{g \in G}(x)$ is an element of R. Let

$$J_i = \{s_i w_x = \sum_{g \in G} s_i(g, x) | x \in E \cup D\} \tag{3.1}$$

for each i and $J = \cup J_i$ in R. For $t \in S$, if $(g_0, x)t = 0$ then $(w_x)t = \sum_{g \in G}(g, x)t = 0$ and if $(g_0, x)t = (h, y)$, for some $y \in E \cup D$ and $h \in G$, then

$$(w_x)t = \sum_{g \in G}(g, x)t = \sum_{h \in G}(h, y) = w_y \tag{3.2}$$

because $G = Gh$. In addition, $w_e w_x = \gamma w_x$ with $\gamma = |G|$ and $e \in E$. Consequently, each $\tilde{J}_i = R(F, J_i)$ is a right ideal and $\tilde{J} = R(F, J)$ is an ideal of R.

LEMMA 3.3. If R is semisimple Artinian, then $\tilde{J}_i$ is a minimal right ideal of $\tilde{J}$ such that $\tilde{J}_i \cong \tilde{J}_j$ for all i and j and $\tilde{J} \cong \oplus \tilde{J}_i$.

PROOF. Suppose $\tilde{A}$ is a nonzero right ideal of $\tilde{J}$ contained in $\tilde{J}_i$. Find a nonzero element $\tilde{a} \in \tilde{A}$ so that $\ell = \ell(\tilde{a})$ in $\tilde{A}$ with respect to the basis J_i is minimal. Suppose $\ell > 1$ and write $\tilde{a} = \sum_{\alpha, x} \alpha s_i w_x$. Then for any j and any $y \in E \cup D$,

$$\tilde{a} s_j w_y = \sum_{\alpha, x} \alpha s_i w_x s_j w_y \in \tilde{A} \tag{3.3}$$

must be 0 otherwise $\tilde{a}s_j w_y = \beta s_i w_y$ has length 1 in $\tilde{A}$ contradicting $\ell > 1$. So $\tilde{a}\tilde{J} = 0$. But since R is semisimple, so is $\tilde{J}$. Then $\tilde{a} = 0$, which is against the choice of $\tilde{a}$. So $\ell = 1$ and then $\tilde{a} = \alpha s_i w_x$ for some $\alpha \in F\backslash\{0\}$ and $x \in E \cup D$. Since there exists $t \in S$ satisfying $(g_0, x)t \in B$; i.e. $(g_0, x)t = (h, e)$ for some $h \in G$, and $e \in E$, we obtain

$$\begin{aligned} \tilde{a}(\alpha^{-1}\gamma^{-1}tw_y) &= (\alpha s_i w_x)(\alpha^{-1}\gamma^{-1}tw_y) = \gamma^{-1}s_i w_x t w_y \\ &= \gamma^{-1}s_i(w_e w_y) = \gamma^{-1}s_i(\gamma w_y) = s_i w_y. \end{aligned} \tag{3.4}$$

But $t(g_0, y) \in tM = s_j M$ for some j implies $\alpha^{-1}\gamma^{-1}tw_y \in \tilde{J}$. So $s_i w_y \in \tilde{A}$, for all $y \in E \cup D$, and $\tilde{J}_i = \tilde{A}$. That is, $\tilde{J}_i$ is a minimal right ideal of $\tilde{J}$.

Note that $J_i \cap J_j = \emptyset$ for all $i \neq j$ implies $\tilde{J}_i \cap \tilde{J}_j = 0$. By mapping $s_i w_x$ to $s_j w_x$ from $\tilde{J}_i$ to $\tilde{J}_j$ we obtain an isomorphism, hence $\tilde{J}_i \cong \tilde{J}_j$. Also $J = \cup_{i=1}^q J_i$, hence $\tilde{J} \cong \oplus \tilde{J}_i$.

PROPOSITION 3.4. If R is semisimple Artinian, $\tilde{J}$ is a simple ideal of R.

PROOF. Let $\tilde{A}$ be a nonzero ideal of R contained in $\tilde{J}$. For each i, if $\tilde{A} \cap \tilde{J}_i \neq 0$ then $\tilde{A} \cap \tilde{J}_i = \tilde{J}_i$. Given any $0 \neq \tilde{a} = \sum_{\alpha,i,x} \alpha s_i w_x$ in $\tilde{A}$, if $(g_0, y)\tilde{a} = 0$ for all $y \in E \cup D$ then $\tilde{J}\tilde{a} = 0$ and so $\tilde{a} = 0$, contradicting $\tilde{a} \neq 0$. So there exists $y \in E \cup D$ such that

$$0 \neq (g_0, y)\tilde{a} = \sum_{\alpha,i,x} \alpha(g_0, y)s_i w_x = \sum_{\beta_x \in F, x \in E \cup D} \beta_x w_x \in \tilde{A}. \tag{3.5}$$

It follows that $s_j(g_0, y)\tilde{a} \in \tilde{J}_j \cap \tilde{A}$ for each j and so $\tilde{J} = \oplus \tilde{J}_i \subseteq \tilde{A}$. Thus $\tilde{J}$ is simple.

Under the assumption that F is algebraically closed, R is semisimple Artinian implies that $\tilde{J}$ is a matrix ring such that $\tilde{J} \cong Mat_n F$. As was mentioned by Jacobson[4], there exists a set of matrix units $\{e_{ij}\}$ such that $\tilde{J}_i = e_{ii}\tilde{J}$. As we can see, each minimal right ideal $\tilde{J}_i$ is an n-dimensional subspace of $\tilde{J}$ with basis J_i. So $|E \cup D| = n$ and the number of the elements in $\{J_i\}$ is also n.

For each i, let $\tilde{J}_i$ be isomorphic to the ith row-subspace in $Mat_n F$ and use $\cong$ to denote the two correspoonding elements between the two sets. Then we have

$$s_i w_x \cong \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \text{ where } a_k \in F \text{ for each } x \in E \cup D. \tag{3.6}$$

Let us begin by studying the first row. For any $e \in E$, recall that $w_e w_e = \gamma(w_e)$ and suppose

$w_e \cong \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$. Then

$$\gamma w_e \cong \gamma \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$
$$= a_1 \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \Longrightarrow a_1 = \gamma \tag{3.7}$$

As to $d \in D$, we know that $w_d w_d = 0$. So

$$w_d \cong \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \Longrightarrow 0 = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$
$$= a_1 \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \Longrightarrow a_1 = 0 \tag{3.8}$$

We conclude that, for $x \in E \cup D$, $w_x \cong \gamma \begin{pmatrix} \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$ where

$$\lambda_{x1} = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \in D \end{cases}.$$

In general, since $s_i w_e w_x = \gamma s_i w_x$ for each i, given $s_i w_e = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith$ we obtain

$$\gamma s_i w_x \cong \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \cdot \gamma \begin{pmatrix} \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$
$$= a_1 \gamma \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith. \tag{3.9}$$

Consequently, $s_i w_x \cong a_1 \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith$. Suppose there exists $f \in E\backslash\{e\}$ and

$s_i w_f \cong \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith$. Then

$$s_i w_f w_x = \gamma s_i w_x \Rightarrow a_1 \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith = b_1 \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith, \tag{3.10}$$

hence $a_1 = b_1 \neq 0$. Now let $\gamma_i = a_1$. We get $s_i w_x \cong \gamma_i \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith$ for each $x \in E \cup D$. In order to study λ_{xi}, we need to look at two different cases of $s_i(g_0, x)$ for each x and each i.

Case 1. If $s_i(g_0, x) \in C_i$ then $(g_0, x)s_i = 0$ and $w_x s_i w_y = 0$ for all $y \in E \cup D$. Thus

$$\begin{aligned} 0 = w_x s w_y &\cong \gamma \begin{pmatrix} \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \gamma_i \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{y1} & \lambda_{y2} & \dots & \lambda_{yn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \\ &= \gamma \lambda_{xi} \gamma_i \begin{pmatrix} \lambda_{y1} & \lambda_{y2} & \dots & \lambda_{yn} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \end{aligned} \tag{3.11}$$

But γ and γ_i are not zero. So $\lambda_{xi} = 0$.

Case 2. If $s_i(g_0, x) \in B_i$ then $(g_0, x)s_i \in B$ and $(g_0, x)s_i = (h, c)$ for some $h \in G$ and $e \in E$.

So $w_x s_i = w_e$ and $w_x s_i w_y = w_e w_y = \gamma w_y$ for all $y \in E \cup D$. That is,

$$\gamma^2 \begin{pmatrix} \lambda_{y1} & \lambda_{y2} & \dots & \lambda_{yn} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = \gamma \begin{pmatrix} \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \gamma_i \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{y1} & \lambda_{y2} & \dots & \lambda_{yn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \tag{3.12}$$

$$= \gamma \lambda_{x_i} \gamma_i \begin{pmatrix} \lambda_{y1} & \lambda_{y2} & \dots & \lambda_{yn} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Hence $\lambda_{x_i} = \gamma\gamma_i^{-1}$.

Let $\gamma_1 = \gamma$, we obtain our next proposition.

PROPOSITION 3.5. $s_i w_x \cong \gamma_i \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{x1} & \lambda_{x2} & \dots & \lambda_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith$, where $\lambda_{xi} = \gamma(\gamma_i)^{-1}$ if $s_i(g_0, x) \in B_i$; and $\lambda_{xi} = 0$ if $s_i(g_0, x) \in C_i$. Thus, for all $x, y \in E \cup D$, either $\lambda_{x_i} = \lambda_{y_i}$ with both $s_i(g_o, x)$ and $s_i(g_0, y)$ are in B_i or $\lambda_{x_i}\lambda_{y_i} = 0$.

With this result, we are ready to find a representation for each element of S. Given $x \in E \cup D$, let

$$h_{x_i} = \begin{cases} g_i, & \text{if } (g_0, x)s_i = (g_i, e) \in B \\ 0, & \text{if } (g_0, x)s_i = 0. \end{cases} \tag{3.13}$$

In particular,

$$h_{x_1} = \begin{cases} g_0, & \text{if } x \in E \\ 0, & \text{if } x \in D \end{cases}. \tag{3.14}$$

Define a mapping $\phi : S \to \mathcal{M}(n, G^0)$ by

$$\phi(s_i(g, x)) = g \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{x1} & h_{x2} & \dots & h_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith. \tag{3.15}$$

ϕ is well-defined for if $s_i(g, x) = s_j(h, y)$ then $i = j$ and $(g, x) = (h, y)$.

PROPOSITION 3.6. S is isomorphic to a left ideal of $\mathcal{M}(n, G^0)$ and, for each i, there exists

$a \in S$ such that

$$\phi(a) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & \dots & g_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \text{ with } g_i \neq 0. \tag{3.16}$$

PROOF. We first claim that ϕ is a monomorphism. By letting $s_i(g,x)$, $s_j(h,y)$ be any two elements in L, we have

$$\phi(s_i(g,x)) = g \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{x1} & h_{x2} & \dots & h_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \text{ and} \tag{3.17}$$

$$\phi(s_j(h,y)) = h \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{y1} & h_{y2} & \dots & h_{yn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} jth. \tag{3.18}$$

So

$$\phi(s_i(g,x))\phi(s_j(h,y)) = gh_{x_j}h \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{y1} & h_{y2} & \dots & h_{yn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith. \tag{3.19}$$

If $(g_0,x)s_j \in B$, then $(g_0,x)s_j = (h_{x_j},e)$ and

$$\begin{aligned} \phi(s_i(g,x)s_j(h,y)) &= \phi(s_i(g,e)(h_{x_j},e)(h,y)) \\ &= \phi(s_i(gh_{x_j}h,y)) \\ &= gh_{x_j}h \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{y1} & h_{y2} & \dots & h_{yn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith. \end{aligned} \tag{3.20}$$

But if $(g_0,x)s_j = 0$ then $h_{x_j} = 0$. In both cases, we see that

$$\phi(s_i(g,x))\phi(s_j(h,y)) = \phi(s_i(g,x)s_j(h,y)). \tag{3.21}$$

Suppose $\phi(s_i(g,x)) = \phi(s_j(h,y))$. Then

$$g\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{x1} & h_{x2} & \dots & h_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith = h\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{y1} & h_{y2} & \dots & h_{yn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} jth. \tag{3.22}$$

First, $i = j$. Next, $gh_{x_k} = hh_{y_k}$ for all k. Then, for each k, either $h_{x_k}, h_{y_k} \in G$ or $h_{x_k} = 0 = h_{y_k}$. Consequently, $\lambda_{x_k} = \lambda_{y_k}$, for all k, and $x = y$ by Proposition 3.5. Thus $g = h$ and $s_i(g,x) = s_j(h,y)$; i.e. ϕ is a monomorphism.

Now we want to show that $\phi(S)$ is a left ideal of $\mathcal{M}(n, G^0)$. Given any $s_i(g,x) \in S$ with

$$\phi(s_i(g,x)) = g\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{x1} & h_{x2} & \dots & h_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \tag{3.23}$$

and any $\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} jth \in \mathcal{M}(n, G^0)$, the product

$$\begin{gathered}\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} jth \cdot g\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{x1} & h_{x2} & \dots & h_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \\ = b_i g\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{x1} & h_{x2} & \dots & h_{xn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} jth\end{gathered} \tag{3.24}$$

is still in $\phi(S)$. Therefore $\phi(S)$ is left ideal of $\mathcal{M}(n, G^0)$.

For each i, there exists $x \in E$ such that $s_i(g_0, x) \in B_i$, hence $(g_0, x)s_i = (h_{x_i}, e)$ for some $e \in E$. Thus $\phi(s_i(g_0,x))$ is an element in $\phi(S)$ whose iith entry is nonzero.

In order to show that R is semisimple Artinian, let us assume the following on a 0-simple semigroup S:

(i) S is finite,

(ii) S is isomorphic to a left ideal of an $n \times n$ mono-row matrix semigroup over a finite group G, denoted by $\mathcal{M} = \mathcal{M}(n, G^0)$, such that for each i, there exists an element $a \in S$ with

$$a \cong \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & \cdots & g_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \text{ and } g_i \neq 0,$$

(iii) the characteristic of F does not divide $|G|$.

By assumption(iii), $\tilde{G} = R(F, G)$ is a semisimple Artinian ring. Then it is stated in [2] that $\tilde{G}$ is the direct sum of its minimal left ideals which are generated by a set of orthorgonal idempotents $\{f_1, f_2, \cdots f_p\}$ and the identity $1 = f_1 + f_2 + \cdots + f_p$. Note that $\tilde{\mathcal{M}} = R(F, \mathcal{M}) = Mat_n(\tilde{G})$. Let

$$(f_i)_{jj} = \begin{array}{c} \qquad\quad jth \\ \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & f_i & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \end{array} jth \text{ for } 1 \leq i \leq p \text{ and } 1 \leq j \leq n. \tag{3.25}$$

Then $\{(f_i)_{jj} | i = 1, 2, ..., p; j = 1, 2, ..., n\}$ is a set of orthorgonal idempotents in $\tilde{\mathcal{M}}$ such that $\sum_{i,j} (f_i)_{jj}$ is equal to the identity matrix in $\tilde{\mathcal{M}}$.

LEMMA 3.7. $\tilde{\mathcal{M}}(f_i)_{jj}$ is a minimal left ideal of $\tilde{\mathcal{M}}$ for each i and j.

PROOF. For each i and j, the left ideal

$$\tilde{\mathcal{M}}(f_i)_{jj} = \{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} (f_i)_{jj} | a_{kl} \in \tilde{G} \text{ for each } k \text{ and } l\}$$

$$= \{ \begin{array}{c} jth \\ \begin{pmatrix} 0 & \dots & a_{1j}f_i & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & a_{nj}f_i & \dots & 0 \end{pmatrix} \end{array} | a_{kj} \in \tilde{G}, k = 1, ..., n\}. \tag{3.26}$$

Also

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{array}{c} jth \\ \begin{pmatrix} 0 & \dots & b_{1j}f_i & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & b_{nj}f_i & \dots & 0 \end{pmatrix} \end{array} = \begin{array}{c} jth \\ \begin{pmatrix} 0 & \dots & a_{1j}b_{jj}f_i & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & a_{nj}b_{jj}f_i & \dots & 0 \end{pmatrix} \end{array}, \tag{3.27}$$

for all $\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in \tilde{\mathcal{M}}$. Since $\tilde{G}f_i$ is a minimal left ideal of $\tilde{G}$, either $\tilde{G}b_{jj}f_i = \tilde{G}f_i$ or $\tilde{G}b_{jj}f_j = 0$. But if $\tilde{G}b_{jj}f_j = 0$ then $b_{jj}f_i = 0$. So $\tilde{\mathcal{M}} \begin{array}{c} jth \\ \begin{pmatrix} 0 & \dots & b_{1j}f_i & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & b_{nj}f_i & \dots & 0 \end{pmatrix} \end{array}$ is either 0 or $\tilde{\mathcal{M}}(f_i)_{jj}$ for any $\begin{array}{c} jth \\ \begin{pmatrix} 0 & \dots & b_{1j}f_i & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & b_{nj}f_i & \dots & 0 \end{pmatrix} \end{array} \in \tilde{\mathcal{M}}(f_i)_{jj}$. That is, $\tilde{\mathcal{M}}(f_i)_{jj}$ is a minimal left ideal of $\tilde{\mathcal{M}}$.

Let $e_{ii} = \begin{array}{cc} ith & \\ \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} & \begin{matrix} \\ \\ ith \\ \\ \\ \end{matrix} \end{array}$ for each i. Then $\tilde{\mathcal{M}} = \oplus \tilde{\mathcal{M}}(f_j)_{ii}$ because

$$\tilde{\mathcal{M}}e_{ii} = \tilde{\mathcal{M}}(f_1)_{ii} \oplus \cdots \oplus \tilde{\mathcal{M}}(f_p)_{ii} \text{ and} \tag{3.28}$$

$$\tilde{\mathcal{M}} = \tilde{\mathcal{M}}e_{11} \oplus \cdots \oplus \tilde{\mathcal{M}}e_{nn}. \tag{3.29}$$

Therefore $\tilde{\mathcal{M}}$ is semisimple Artinian.

PROPOSITION 3.8. $R \cong \tilde{\mathcal{M}}$.

PROOF. For each i, there exists an element $a \in S$ such that

$$a \cong \begin{array}{cc} \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & \dots & g_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} & \begin{matrix} \\ \\ ith \\ \\ \\ \end{matrix} \end{array} \tag{3.30}$$

and $g_i \neq 0$ by assumption (ii). Then

$$(e_{ii}) = \begin{array}{cc} \begin{array}{c} ith \end{array} & \\ \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & g_i^{-1} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} & ith \end{array} \cdot a \in R(F, \phi(S)), \tag{3.31}$$

hence $0 \neq R(F, \phi(S))(f_j)_{ii} = \tilde{\mathcal{M}}(f_j)_{ii}$. It follows that $R \cong \tilde{\mathcal{M}}$.

With all the properties found here, we obtain the major result,

THEOREM 3.9. Let F be an algebraically closed field and S be a 0-simple semigroup. Then R is semisimple Artinian if and only if the following hold:

(1) S is finite;

(2) S is isomorphic to a left ideal of a mono-row matrix semigroup $\mathcal{M}(n, G^0)$ where $G = eSe$ and e is an idempotent of S, such that for each i, there is an element $a \in S$ with

$$a \cong \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & \cdots & g_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \text{ and } g_i \neq 0; \tag{3.32}$$

(3) The characteristic of F does not divide $|G|$.

According to Theorem 5.20 in [1], the sandwich matrix P in the regular Rees matrix semigroup, $\mathcal{M}(G; h, k; P)$, is nonsingular (in particular, $h = k$). Here $h = k$ means the number of distinct 0-minimal right (and left) ideals of S. So the size of P is the same as that of the mono-row matrices found in this part.

Some readers may find that the representation described in Theorem 3.9 is a special case of the dual Schutazenberger representation mentioned in [1]. But the approaches are different.

4. RELATION TO REES MATRIX REPRESENTATION.

Hereafter, we would like to check if we can find the Rees matrix representation from the Main Theorem directly. First let us look at the left ideals generated by elements from $E \cup D$.

LEMMA 4.1. $S(g_0, x) \neq S(g_0, y)$ for all distinct $x, y \in E \cup D$.

PROOF. Suppose there exist $x, y \in E \cup D$ such that $S(g_0, x) = S(g_0, y)$. Then

$$(g_0, x) = (g_0, e)(g_0, x) \in S(g_0, x) \Longrightarrow (g_0, x) = s(g_0, y) \tag{4.1}$$

for some $s \in S$. $s \in M$ for if $sM \neq M$ then $(g_0, x) \notin M$ is a contradiction. If $s \in C$ then $(g_0, x) = s(g_0, y) = 0$ which causes another contradiction. So s must be in B and $s = (g, f)$ for some $g \in G$ and $f \in E$. Hence

$$(g_0, x) = (g, f)(g_0, y) = (g, y) \Longrightarrow g_0 = g \text{ and } x = y. \tag{4.2}$$

Since the set $E \cup D$ is finite, we can list the elements in an order with one of the element $e \in E$ to be the first. Using the same notations in [1], let $(g_0, x_\lambda) = q_\lambda$, $s_i(g_0, e) = r_i$, and

$$p_{\lambda i} = \begin{cases} q_\lambda r_i, & \text{if } q_\lambda r_i \in H_{11} \\ 0, & \text{otherwise} \end{cases} . \tag{4.3}$$

The lemma above helps us obtaining the nonsingular sandwich matrix $P = (p_{\lambda i})$ over H_{11}^0(which is the same as G^0). Note that for each i, $p_{\lambda i} = (g_0, x_\lambda)s_i(g_0, e) = h_{x_\lambda i}$. So

$$P = \begin{pmatrix} h_{e1} & h_{e2} & \dots & h_{en} \\ \vdots & \vdots & \ddots & \vdots \\ h_{x_\lambda 1} & h_{x_\lambda 2} & \dots & h_{x_\lambda n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{x_n 1} & h_{x_n 2} & \dots & h_{x_n n} \end{pmatrix} \tag{4.4}$$

and for each $s \in S$

$$\begin{aligned} s = g & \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{x_\lambda 1} & h_{x_\lambda 2} & \dots & h_{x_\lambda n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} ith \qquad (4.5) \\ &= s_i(g, x_\lambda) = s_i(g_0, e)(g, e)(g_0, x_\lambda) \\ &= r_i(g, e)q_\lambda = (g)_{i\lambda}; \text{ the Rees matrix.} \end{aligned}$$

This shows the relation between the Rees matrix and the matrix described in this article.

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