NON-ARCHIMEDEAN EBERLEIN-ŠMULIAN THEORY

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ABSTRACT. It is shown that, for a large class of non-archimedean normed spaces E , a subset X is weakly compact as soon as $f(X)$ is compact for all $f \in E'$ (Theorem 2.1), a fact that has no analogue in Functional Analysis over the real or complex numbers. As a Corollary we derive a non-archimedean version of the Eberlein-Šmulian Theorem (2.2 and 2.3, for the 'classical' theorem, see [1], VIII, §2 Theorem 1 and Corollary, page 219).

KEY WORDS AND PHRASES. Non-archimedean Banach space, weak compactness.

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INTRODUCTION

Let E be a two-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ and let $X := \{x \in E : 0 < \|x\| \leq 1\}$. Each $f \in E'$ has zeros on X , so $f(X) = f(\{0\} \cup X)$ is compact, while obviously X is not. The same story can be told when we replace $\mathbb{R}$ or $\mathbb{C}$ by a complete non-trivially valued non-archimedean field K that is locally compact. However, if K is not locally compact then, under reasonable conditions, for a subset X of a normed space E over K compactness of $f(X)$ for all $f \in E'$ implies weak compactness of X (we point out that if such an X has more than one point it cannot be convex). To prove this curious fact (in §2) we shall develop some machinery in §1.

PRELIMINARIES

Throughout K is a non-trivially non-archimedean valued field which is complete with respect to the metric induced by the valuation $|\cdot|$, and E is a normed K -vector space, where we assume $\|\cdot\|$ to satisfy the strong triangle inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$. We write $|K^\times| := \{|\lambda| : \lambda \in K, \lambda \neq 0\}$, $B_E(0, r) := \{x \in E : \|x\| \leq r\}$, $B_E := B_E(0, 1)$.

E' is the space of all linear continuous functions $E \rightarrow K$. Equipped with the norm $f \mapsto \sup\{|f(x)| : x \in B_E\}$ it is a Banach space (i.e. a complete normed space). E is called *normpolar* if the norm is polar i.e. if $\|x\| = \sup\{|f(x)| : f \in E', |f| \leq \|\cdot\|\} \quad (x \in E)$, in other words, if $j : E \rightarrow E''$ is an isometry. E' is always normpolar. We assume throughout this note that E is normpolar.

A subset A of a (normed) space E is *absolutely convex* if it is a module over B_K . A set $X \subset E$ is *convex* if it is either empty or an additive coset of an absolutely convex set. A subset A of E is called *edged* if it is absolutely convex and, in case the valuation of K is dense, $A = \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. The *weak topology* $w = \sigma(E, E')$ is the weakest topology on E making all $f \in E'$ continuous. The *weak-star topology* $w' = \sigma(E', E)$ is the weakest topology

on E' making all evaluation maps $f \mapsto f(a)$ ($a \in E$) continuous. For $X \subset E'$ we denote its w' -closure by $\overline{X}^{w'}$.

For other notions used in this paper we refer to [4].

1. SEPARATION OF w' -PRECOMPACT SETS

LEMMA 1.1. *Let X be a bounded subset of E' . Then $\{x \in E : \inf_{f \in X} |f(x)| > 0\}$ is open in E .*

Proof. X is equicontinuous, so for each $n \in \mathbb{N}$ the set $U_n := \{x \in E : |f(x)| > \frac{1}{n} \text{ for all } f \in X\}$ is open. Then so is $\bigcup_n U_n = \{x \in E : \inf_{f \in X} |f(x)| > 0\}$.

LEMMA 1.2. *Let K be not locally compact. Let $X \subset E'$ and $a \in E$ be such that $X(a) := \{f(a) : f \in X\}$ is precompact. Suppose $X \subset g + U$ where U is an edged zero neighbourhood in E' , U w' -closed and where $g \in E' \setminus U$. Then for any $\varepsilon > 0$ there exists a $b \in E$ for which $\|a - b\| \leq \varepsilon$ and $\inf\{|f(b)| : f \in X\} > 0$.*

Proof. There exists an $r \in |K^\times|$ such that $B_{E'}(0, r) \subset U$. Choose $\delta \in K, 0 < |\delta| < 1$. The equivalence relation $\sim$ on $K^\times$ given by ' $\alpha \sim \beta$ iff $|\alpha - \beta| < |\beta|$ ' yields an open partition of $C := \{\lambda \in K : |\delta|r\varepsilon \leq |\lambda| \leq r\varepsilon\}$ that is infinite because K is not locally compact. By precompactness $X(a)$ cannot meet each equivalence class and there exists a $\gamma \in C$ such that

$$(*) \quad |f(a) - \gamma| \geq |\gamma| \quad (f \in X).$$

U is w' -closed and edged, $g \notin U$, so by [6], 4.8 there exists a $c \in E$ such that $g(c) = \gamma, |f(c)| < |\gamma|$ for all $f \in U$. Set $b := a - c$. We have $|f(c)| \leq |\gamma|$ for all $f \in B_{E'}(0, r)$ so $\|a - b\| = \|c\| = \|g(c)\| \leq |\gamma|r^{-1} \leq \varepsilon$. For each $f \in X$, writing $f = g + u$ where $u \in U$, we obtain $|f(c) - \gamma| = |f(c) - g(c)| = |u(c)| < |\gamma|$. This, combined with $(*)$, yields $|f(a) - \gamma| > |f(c) - \gamma|$ for all $f \in X$, so $|f(b)| = |f(a) - f(c)| = \max(|f(a) - \gamma|, |f(c) - \gamma|) = |f(a) - \gamma| \geq |\gamma|$. It follows that $\inf_{f \in X} |f(b)| > 0$.

COROLLARY 1.3. *Let K be not locally compact, let E be a Banach space. Let $X \subset E'$ be w' -precompact. Suppose $X \subset g + U$ where U is an edged zero neighbourhood in E' , U w' -closed, $g \in E' \setminus U$. Then $\{x \in E : \inf_{f \in X} |f(x)| > 0\}$ is open and dense in E .*

Proof. Just combine Lemmas 1.1 (w' -precompactness implies w' -boundedness hence norm boundedness by completeness) and 1.2.

DEFINITION 1.4. Let us call $X \subset E'$ σ -decomposable in E' if for each $g \in E' \setminus X$ there exist $f_1, f_2, \dots \in X$ and edged zero neighbourhoods $U_1, U_2, \dots$ in E' such that each U_n is w' -closed and $X \subset \bigcup_n (f_n + U_n)$, $g \notin \bigcup_n (f_n + U_n)$.

THEOREM 1.5. (SEPARATION THEOREM) *Let K be not locally compact, let E be a Banach space, let $X \subset E'$ be w' -precompact and σ -decomposable in E' . Then for each $g \in E' \setminus X$ there exists an $a \in E$ such that $g(a) \neq f(a)$ for all $f \in X$.*

Proof. Without loss, assume $g = 0$. Let $\{f_n + U_n : n \in \mathbb{N}\}$ be a covering of X like in Definition 1.4. By Corollary 1.3 for each $n \in \mathbb{N}$ the set $\{x \in E : \inf_{f \in X_n} |f(x)| > 0\}$ is open and dense in E , where $X_n := X \cap (f_n + U_n)$. By completeness and the Baire Category Theorem $\{x \in E : f(x) \neq 0 \text{ for all } f \in X\} \supset \bigcap_n \{x \in E : \inf_{f \in X_n} |f(x)| > 0\} \neq \emptyset$.

REMARK. It is not hard, by modifying 1.1 - 1.5, to prove the following dual form of this separation theorem. Let K be not locally compact, let $X \subset E$ be weakly precompact and σ -decomposable in E (see below). Then for each $a \in E \setminus X$ there exists an $f \in E'$ such that $f(a) \notin f(X)$. Here, X is called σ -decomposable in E if for each $a \in E \setminus X$ there exist $x_1, x_2, \dots \in X$ and edged zero neighbourhoods $U_1, U_2, \dots$ in E such that each U_n is weakly closed and $X \subset \bigcup_n (x_n + U_n)$, $a \notin \bigcup_n (x_n + U_n)$.

COROLLARY 1.6. *Let K be not locally compact, let E be a Banach space, let $X \subset E'$ be σ -decomposable in E' . Suppose $X(a) := \{f(a) : f \in X\}$ is compact for all $a \in E$. Then X is w' -compact.*

Proof. The map $f \mapsto (f(a))_{a \in E}$ is a homeomorphism of (E', w') onto a subspace of K^E . The image of X lies in the compact subset $\prod_{a \in E} X(a)$ so X is w' -precompact. Since E' is w' -quasicomplete by the p-adic Alaoglu Theorem [8], 3.1, it suffices to show that X is w' -closed. To this end, let $g \in E' \setminus X$. By Theorem 1.5 there exists an $a \in E$ such that $g(a) \notin X(a)$. Now $X(a) \subset \overline{X^{w'}(a)} \subset \overline{X(a)} = X(a)$, so $g(a) \notin \overline{X^{w'}(a)}$ i.e. $g \notin \overline{X^{w'}}$.

To find examples of σ -decomposable sets (in 1.9-1.11) we need the following Lemmas.

LEMMA 1.7. *Let $n \in \mathbb{N}$, let D be an n -dimensional subspace of E' . Then for each $t \in (0, 1)$ there exist $a_1, a_2, \dots, a_n \in B_E$ such that $\max_{1 \leq i \leq n} |f(a_i)| \geq t \|f\|$ ($f \in D$).*

Proof. First assume that the valuation of K is dense. The space $H := \{x \in E : f(x) = 0 \text{ for all } f \in D\}$ has codimension n in E . Choose $s \in (t, 1)$ and let $g_1, \dots, g_n$ be a $\sqrt{s}$ -orthogonal base of $(E/H)'$ such that $s^{-1} \leq \|g_i\| \leq t^{-1}$ for $i \in \{1, \dots, n\}$. There exist $b_1, \dots, b_n \in E/H$ such that $g_i(b_j) = \delta_{ij}$ ($i, j \in \{1, \dots, n\}$). Let $i \in \{1, \dots, n\}$, let $g = \sum \lambda_j g_j \in (E/H)'$. Then $\|g\| \geq \sqrt{s} \max |\lambda_j| \|g_j\|$ and $|g(b_i)| = |\lambda_i|$ so $|g(b_i)| \leq \max_j |\lambda_j| \leq s \max_j |\lambda_j| \|g_j\| \leq \sqrt{s} \|g\|$. So $\|b_i\| < 1$. Thus, with $\pi : E \rightarrow E/H$ denoting the canonical quotient map, there exist $a_1, \dots, a_n \in B_E$ with $\pi(a_i) = b_i$ for each i . The adjoint π' of π maps $(E/H)'$ isometrically onto D . Now let $f \in D$. Then $f = \pi'(g)$ where $g \in (E/H)'$, $\|g\| = \|f\|$. We have, writing $g = \sum_{j=1}^n \lambda_j g_j$, $\max_{1 \leq i \leq n} |f(a_i)| = \max_i |g(b_i)| = \max_i |\lambda_i| \geq t \max_i |\lambda_i| \|g_i\| \geq t \|\sum \lambda_i g_i\| = t \|g\| = t \|f\|$.

Now, if the valuation is discrete we can modify the above proof by taking $s = t = 1$. Then the b_i have norm ≤ 1 (rather than < 1), but one can use that E/H is a strict quotient i.e. there exist $a_1, \dots, a_n \in E$ with $\|a_i\| = \|b_i\|$ and $\pi(a_i) = b_i$ for each i .

LEMMA 1.8. *Let D be a subspace of E' , D of countable type. Then there is a sequence $a_1, a_2, \dots \in B_E$ such that $\|f\| = \sup_n |f(a_n)|$ for all $f \in D$.*

Proof. Let $D_1 \subset D_2 \subset \dots$ be finite-dimensional subspaces of D , $\bigcup_n D_n$ is dense in D . Let $t \in (0, 1)$. By Lemma 1.7 there exists a finite set $F_n^t \subset B_E$ such that $\max_{x \in F_n^t} |f(x)| \geq t \|f\|$ for all $f \in D_n$.

So, for $F^t := \bigcup_{n \in \mathbb{N}} F_n^t$ we obtain

$$(*) \quad \|f\| \geq \sup_{x \in F^t} |f(x)| \geq t \|f\| \quad (f \in \bigcup_n D_n).$$

Now $F := \bigcup_{t \in \mathbb{Q} \cap (0, 1)} F^t$ is countable and $(*)$ implies $\|f\| = \sup_{x \in F} |f(x)|$ for all $f \in \bigcup_n D_n$, hence, by continuity, for all $f \in D$.

PROPOSITION 1.9. *Let $X \subset E'$ be such that $X(a) := \{f(a) : f \in X\}$ is separable for each $a \in E$ and $[X]$ is of countable type. Then X is σ -decomposable in E' .*

Proof. Let $g \in E' \setminus X$. Then $D := [\{g\} \cup X]$ is of countable type so by Lemma 1.8 there exist $a_1, a_2, \dots \in B_E$ such that

$$(*) \quad \|h\| = \sup_{n \in \mathbb{N}} |h(a_n)| \quad (h \in D).$$

For each $m, n \in \mathbb{N}$ the set $U_{mn} := \{h \in E' : |h(a_n)| \leq \frac{1}{m}\}$ is an edged w' -zero neighbourhood. Its cosets, except for $g + U_{mn}$, cover $X \setminus (g + U_{mn})$ and by separability of $X(a_n)$ there exists a countable subcovering F_{mn} no member of which contains g . Then $\bigcup_{m,n} F_{mn}$ still avoids g ; it remains to be shown that it covers X . Suppose $f \in X$ is not covered. Then $f \in g + U_{mn}$ for all

m, n so $|f(a_n) - g(a_n)| = 0$ for all n . Now $f - g \in D$, so by $(*)$ we obtain $\|f - g\| = 0$ i.e. $f = g$, a contradiction since $g \notin X$.

COROLLARY 1.10. *Let $X \subset E'$. If X is norm precompact, or X is w' -precompact and $[X]$ is of countable type, then X is σ -decomposable in E' .*

PROPOSITION 1.11. *Let $X \subset E'$ be such that $X(a)$ is separable for each $a \in E$. Suppose that for each $h \in \overline{X}^{w'}$ the set $X \cup \{h\}$ is w' -metrizable. Then X is σ -decomposable in E' .*

Proof. Let $g \in E' \setminus X$. If $g \notin \overline{X}^{w'}$ then there exists a w' -zero neighbourhood U such that $(g + U) \cap X = \emptyset$. We may assume that U is of the form $\{f \in E' : |f(a_1)| \leq \varepsilon, \dots, |f(a_n)| \leq \varepsilon\}$ for some $\varepsilon > 0$, $n \in \mathbb{N}$, $a_1, \dots, a_n \in E$. Then U is w' -closed and edged. By separability of $X(a_1) \times \dots \times X(a_n)$ only countably many of the cosets $f + U : f \in X$ cover X and none of them contains g . Now let $g \in \overline{X}^{w'}$. By w' -metrizability there exist w' -neighbourhoods of zero $U_1 \supset U_2 \supset \dots$ such that $X \cap \bigcap_n (g + U_n) = \emptyset$. We may suppose that the U_n are w' -closed and edged. By separability, like above, for each n the set $X \setminus (g + U_n)$ is covered by countably many additive cosets of U_n none of them containing g . Their union is a countable covering of X avoiding g .

2. EBERLEIN-ŠMULIAN THEORY

We now apply the theory of §1. Recall ([5], p. 57) that E is said to have property $(*)$ if for each subspace D of countable type, every $f \in D'$ has an extension $\tilde{f} \in E'$. By the non-archimedean Hahn-Banach Theorem [4], 4.8 every normed space over a spherically complete K has $(*)$. For general K , spaces with a base, in particular spaces of countable type, have $(*)$ ([5], p. 58), and so have strongly polar spaces ([6], 4.2). Recall that E is assumed to be normpolar.

THEOREM 2.1. *Let K be not locally compact, let X be a subset of E such that $f(X)$ is compact for all $f \in E'$. Then each one of the following properties implies that X is weakly compact and weakly metrizable.*

- (i) E has property $(*)$.
- (ii) E' is of countable type.
- (iii) $[X]$ is of countable type.

Moreover, in case (i) X is norm compact and the weak and norm topology coincide on X .

Proof. The natural isometry $j : E \rightarrow E''$ is easily seen to be a homeomorphism of E with the weak topology onto $j(E)$ with the restriction of the w' -topology $\sigma(E'', E')$. We show that $j(X)$ is σ -decomposable in E'' . First note that the predual E' is normpolar. In case (i), from weak precompactness of X it follows that X is norm precompact by [7], Th. 3 (the assumption made throughout [7] that E is complete is easily seen to be superfluous here). So $j(X)$ is norm precompact in E'' and therefore σ -decomposable by Corollary 1.10. For case (ii) observe that every $(w'-)$ bounded subset of E'' is w' -metrizable ([8], 6.1) which applies to $j(X) \cup \{\theta\}$ for any $\theta \in E''$. For each $f \in E'$ the set $j(X)(f) = f(X)$ is compact hence separable so $j(X)$ is σ -decomposable in E'' by Proposition 1.11. For case (iii) we can directly apply Corollary 1.10. Thus, $j(X)$ is σ -decomposable, and from Corollary 1.6 we conclude that $j(X)$ is w' -compact, so $X = j^{-1}(j(X))$ is w -compact. Observe that X is w -bounded hence bounded by normpolarity ([6], 7.7).

We have seen in passing that $j(X)$ is w' -metrizable in case (ii), so X is weakly metrizable. Now let X satisfy (iii). Then $[j(X)]$ is of countable type so by Lemma 1.8 there exist $f_1, f_2, \dots \in B_{E'}$ such that $\|j(x)\| = \sup_{n \in \mathbb{N}} |f_n(x)|$ for all $x \in X$. The formula $d(x, y) = \sup_n |f_n(x) - f_n(y)| 2^{-n}$ defines an ultrametric d on X (if $d(x, y) = 0$ then $|f_n(x) - f_n(y)| = 0$ for all n so $\|x - y\| = 0$). By boundedness of X the induced topology is weaker than the weak topology on X , but by

weak compactness these topologies coincide and so X is weakly metrizable. Finally, in case (i) apply [6], 5.12 to conclude that on X the weak and norm topology coincide, and that therefore X is norm compact and w -metrizable.

REMARKS.

1. If K is not spherically complete the space ℓ^∞ does not have property $(*)$ ([4], 4.15 $(\delta) \Rightarrow (\gamma)$) but since $(\ell^\infty)' \simeq c_0$ ([4], 4.17) it satisfies (ii) of the above Theorem, and so do the non-reflexive space $\ell^\infty \widehat{\otimes} \ell^\infty$ ([3], 2.3) and the space D of [4], 4.J.
2. Let K be not spherically complete, let $E := \ell^\infty$, let $X := \{0\} \cup \{e_1, e_2, \dots\} \subset \ell^\infty$, when $e_1, e_2, \dots$ are the unit vectors. Then (ii) and (iii) above hold. X is weakly compact (since $\lim_{n \rightarrow \infty} e_n = 0$ weakly) but is obviously not norm compact.
3. The following example indicates that extending Theorem 2.1 to, say, metrizable locally convex spaces is doubtful. Let $E := K^{\mathbb{N}}$ with the product topology. Then $E' \cong \bigoplus_{\mathbb{N}} K$. Let $X := \{e_1, e_2, \dots\}$ where $e_1, e_2, \dots$ are the unit vectors of $K^{\mathbb{N}}$. Then E is of countable type so (i), (ii), (iii) of Theorem 2.1 are (formally) satisfied. For each $f \in E'$ we have $f(e_n) = 0$ for large n , so $f(X)$ is finite (hence compact) and contains 0. Yet, X is not (weakly) compact as $0 = w - \lim_{n \rightarrow \infty} e_n \notin X$.

The following is now an almost trivial consequence of Theorem 2.1.

COROLLARY 2.2. (p-adic Eberlein-Šmulian Theorem I) *Let K be not locally compact and let X, E satisfy one of the conditions (i), (ii), (iii) of Theorem 2.1. Then the following are equivalent.*

- (α) X is weakly compact.
- (β) X is weakly sequentially compact.
- (γ) X is weakly countably compact.

Proof. Each one of the properties (α), (β), (γ) implies compactness of $f(X)$ for all $f \in E'$. By Theorem 2.1 X is weakly metrizable and from that the equivalence of (α), (β), (γ) follows easily.

NOTE. In Corollary 2.2, (α), (β), (γ) are obviously equivalent to: 'for all $f \in E'$ the image $f(X)$ is compact.'

We have seen in the Introduction that Theorem 2.1 fails if K is locally compact. We now investigate what happens to Corollary 2.2. Note that every normed space over K has $(*)$.

THEOREM 2.3. (p-adic Eberlein-Šmulian Theorem II) *Let K be locally compact, let $X \subset E$. Then each one of the above statements (α), (β), (γ) is equivalent to ' X is norm compact'.*

Proof. We have $(\alpha) \Rightarrow (\gamma)$, $(\beta) \Rightarrow (\gamma)$. It suffices to prove that (γ) implies that X is a norm compactoid (then X is weakly metrizable since the norm and weak topology coincide on X ([6], 5.12)). Suppose not. Then by [7], Th. 2 there is a $t \in (0, 1]$ and a t -orthogonal sequence $e_1, e_2, \dots$ in X such that $\inf_n \|e_n\| > 0$. By (γ) there is a weak accumulation point a of $\{e_1, e_2, \dots\}$. This a is in the weak closure D of $\llbracket e_1, e_2, \dots \rrbracket$ which equals the norm closure, so $a = \sum_{i=1}^{\infty} \lambda_i e_i$, where $\|\lambda_i e_i\| \rightarrow 0$. If $\lambda_j \neq 0$ for some j , let $U := \{x \in E : |\delta_j(x)| < |\lambda_j|\}$ where $\delta_j \in E'$ is an extension of the j th coordinate function $\Sigma \xi_i e_i \mapsto \xi_j$ on D . Then $a + U$ is a weak neighbourhood of a but for each $n \in \mathbb{N}$, $n \neq j$ we have $|\delta_j(a - e_n)| = |\lambda_j|$ so $e_n \notin a + U$, a contradiction. Hence, $a = 0$. But then $\{x \in E : |f(x)| < 1\}$ is a weak neighbourhood of a containing no e_n if $f \in E'$ is such that $f(e_n) = 1$ for all n . Contradiction, so X is a norm compactoid.

REMARK. Corollary 2.2 for strongly polar spaces E and Theorem 2.3 were first proved directly by the first author.

REMARK. The following 'relative' version of the Eberlein-Šmulian Theorem holds. (Compare [1], VIII §2, Theorem 1). Let $X \subset E$. Suppose one of the conditions (i), (ii), (iii) of Theorem 2.1 is satisfied. Then the following are equivalent. (α) X is weakly relatively compact. (β) X is weakly relatively sequentially compact. (γ) X is weakly relatively countably compact. We leave the easy proof to the reader.

COUNTEREXAMPLES. We show that the previous theory fails for certain subsets X of $\ell^\infty(I)$ where I has at least the cardinality of the continuum, but is non-measurable, and where K is not spherically complete. The K -valued characteristic function of a subset $S \subset I$ is denoted ξ_S and is given by $\xi_S(x) := 1$ if $x \in S$, $\xi_S(x) := 0$ if $x \in I \setminus S$.

1. Let $X := \{\xi_S : S \subset I\}$. Then X is a weakly compact but not weakly sequentially compact subset of $\ell^\infty(I)$.

Proof. X is bounded and since $\ell^\infty(I)' \simeq c_0(I)$ ([4], 4.21) the weak topology on X is the topology of pointwise convergence. Clearly the map $f \mapsto (f(i))_{i \in I}$ is a homeomorphism $X \rightarrow \{0, 1\}^I$, hence X is weakly compact. To prove that X is not weakly sequentially compact, let $\phi : I \rightarrow Y$ be a surjection where $Y := \{\xi_A : A \subset \mathbb{N}\} \subset \ell^\infty$. The formula $\phi(x) = (\xi_{S_1}(x), \xi_{S_2}(x), \dots)$ ($x \in I$) defines subsets $S_1, S_2, \dots$ of I . If $\xi_{S_{n_1}}, \xi_{S_{n_2}}, \dots$ is a subsequence of $\xi_{S_1}, \xi_{S_2}, \dots$ then, by surjectivity of ϕ , there is an $x \in I$ for which $(\xi_{S_{n_1}}(x), \xi_{S_{n_2}}(x), \dots) = (1, 0, 1, 0, 1, \dots)$, so the subsequence is not weakly convergent.

2. Let $Z := \{\xi_S : S \subset I, S \text{ countable}\} \subset \ell^\infty(I)$. Then Z is weakly sequentially compact but not weakly compact.

Proof. Clearly the weak closure of Z equals X of above, so Z is not weakly compact. On the other hand, if $\xi_{S_1}, \xi_{S_2}, \dots$ is a sequence in Z then $S := \cup S_n$ is countable and by a standard diagonal procedure one obtains a subsequence converging at all points of S , hence at all points of I , to an element of Z .

3. Let $T := \{\xi_{\{i\}} : i \in I\} \subset \ell^\infty(I)$. Then $f(T)$ is compact for all $f \in \ell^\infty(I)'$ but T is not weakly countably compact.

Proof. Let $f \in \ell^\infty(I)'$. As $\ell^\infty(I)' \simeq c_0(I)$ we have that $f(\xi_{\{i\}}) = 0$ except for $i \in \{i_1, i_2, \dots\}$ where we may assume the $i_n \in I$ to be distinct. Then $\xi_{\{i_n\}} \rightarrow 0$ weakly so $T_1 := \{0\} \cup \{\xi_{\{i_n\}} : n \in \mathbb{N}\}$ is weakly compact and $f(T) = f(T_1)$ is compact. However the only weak accumulation point of $\{\xi_{\{i_1\}}, \xi_{\{i_2\}}, \dots\}$ is $0 \notin T$ so that T is not weakly countably compact.

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