ON A STRUCTURE SATISFYING $F^K - (-)^{K+1}F = 0$

LOVEJOY S. DAS

Department of Mathematics
Kent State University
Tuscarawas Campus
New Philadelphia, OH 44663

(Received October 23, 1993 and in revised form July 6, 1994)

ABSTRACT. In this paper we shall obtain certain results on the structure defined by $F(K, -(-)^{K+1})$ and satisfying $F^K - (-)^{K+1}F = 0$, where F is a non null tensor field of the type (1,1). Such a structure on an n -dimensional differentiable manifold M^n has been called $F(K, -(-)^{K+1})$ structure of rank " r ", where the rank of F is constant on M^n and is equal to " r ". In this case M^n is called an $F(K, -(-)^{K+1})$ manifold. The case when K is odd has been considered in this paper.

KEY WORDS AND PHRASES. f -structure, Integrability Conditions, Conformal Diffeomorphism, Nijenhuis Tensor.

AMS SUBJECT CLASSIFICATION CODE. 53C15.

1. INTRODUCTION.

Let F be a non zero tensor field of the type (1,1) and of class C^∞ on M^n such that [2]

$$F^K - (-)^{K+1}F = 0 \quad \text{and} \quad F^\omega - (-)^{\omega+1}F \neq 0 \quad (1.1)$$

for $1 < \omega < K$, where K is a fixed positive integer greater than 2. The degree of the manifold being K , ($K \geq 3$). Let us define operators on M^n by:

$$\tilde{I} \stackrel{\text{def}}{=} (-)^{K+1}F^{K-1}, \quad \tilde{m} \stackrel{\text{def}}{=} I - (-)^{K+1}F^{K-1} \quad (1.2)$$

where I denotes the identity operator on M^n . Thus from (1.1) and (1.2) the following results are obvious

$$\tilde{I} + \tilde{m} = I, \quad \tilde{I}^2 = \tilde{I}, \quad \tilde{m}^2 = \tilde{m}.$$

For F satisfying (1.1), there exists complementary distributions $\tilde{L}$ and $\tilde{M}$, corresponding to the projection operators $\tilde{I}$ and $\tilde{m}$ respectively. Now we state the following theorems [2].

THEOREM (1.1). We have

$$F\tilde{I} = \tilde{I}F = F \quad \text{and} \quad F\tilde{m} = \tilde{m}F = 0 \quad (1.3)$$

THEOREM (1.2). Let the tensor field $F (\neq 0)$ satisfy (1.1) and let the operators $\tilde{I}$ and $\tilde{m}$ defined by (1.2). Then it admits an almost product structure on $\tilde{L}$ and null operator on $\tilde{M}$. That is

$$F^{K-1}\tilde{I} = \tilde{I} \quad \text{and} \quad F^{K-1}\tilde{m} = \tilde{m}F^{K-1} = 0 \quad (1.4)$$

Then $F^{\frac{K-1}{2}}$ acts on $\tilde{L}$ as an almost product structure and on $\tilde{M}$ as a null operator.

THEOREM (1.3). If in M^n there is given a tensor field $F (F \neq 0, F^{K-1} \neq I)$ of type (1,1) and of class C^∞ such that $F^{K-1} - (-)^{K+1} F = 0$, then M^n admits an almost product structure $\overset{\circ}{\Psi} = 2(-)^{K+1} F^{K-1} - I$ where $\overset{\circ}{\Psi} \stackrel{\text{def}}{=} \tilde{l} - \tilde{m}$.

PROOF. We have

$$\begin{aligned}\overset{\circ}{\Psi} &\stackrel{\text{def}}{=} \tilde{l} - \tilde{m}, \\ &= 2(-)^{K+1} F^{K-1} - I\end{aligned}$$

Then

$$\overset{\circ}{\Psi} \neq I \quad \text{if} \quad F^{K-1} \neq I$$

Also,

$$\begin{aligned}\overset{\circ}{\Psi}^2 &= 4(-)^{2K+2} F^{2K-2} + I - 4(-)^{K+1} F^{K-1} \\ &= 4F^K F^{K-2} + I - 4(-)^{K+1} F^{K-1} \\ &= 4F^{K-1} + I - 4F^{K-1}, \quad \text{from (1.1)} \\ &= I.\end{aligned}$$

Thus,

$$\overset{\circ}{\Psi} \neq I \quad \text{if} \quad F^{K-1} \neq I,$$

and

$$\overset{\circ}{\Psi}^2 = I \quad \text{if} \quad F^{K-1} \neq I.$$

Hence $\overset{\circ}{\Psi}$ is an almost product structure.

2. METRIC FOR $F(K, -(-)^{K+1})$ STRUCTURE.

THEOREM (2.1). Let M^n be an $F(K, -(-)^{K+1})$ manifold of degree K defined by $F^K - (-)^{K+1} F = 0$ and $F^\omega - (-)^{\omega+1} F \neq 0$ for $1 < \omega < K$, and K is a fixed positive integer greater than 2, then:

there exists a positive definite Riemannian metric g with respect to which $\tilde{L}$ and $\tilde{M}$ are orthogonal and such that:

$$H_j^i H_1^s g_{ts} + \tilde{m}_j^t g_{ti} = g_{ji},$$

$$H_{\tilde{m}} = H_{ij},$$

where

$$H = F^{\frac{K-1}{2}} \quad \text{and} \quad H_{\tilde{m}} = H_j^i g_{ti}$$

and the rank of F is odd.

PROOF. Let us consider local coordinate system in the manifold M^n and let us denote the local components of the tensor ϕ in the set $\{F, \tilde{l}, \tilde{m}, H\}$ by ϕ_i^p . Here we consider r -mutally orthogonal unit vectors $u_a^p (a, b, c, \dots = 1, 2, 3, \dots, r)$ in $\tilde{L}$ and $(n-r)$ mutually orthogonal unit vectors

$$u_A^p (A, B, C, \dots = r+1, r+2, \dots, n) \quad \text{in} \quad \tilde{M}^n.$$

(ω_i^a, ω_i^A) denotes the inverse matrix of (u_b^p, u_B^p) .

Then ω_i^a and ω_i^A are both components of linearly independent covariant vectors. Let

$$\begin{aligned}
\tilde{m}_{j'} &= \tilde{m}'_j a_{t'} , \\
a_{j'} &= \omega_j^a \omega_{t'}^a + \omega_j^1 \omega_{t'}^1 \\
g_{j'} &= \frac{1}{2} (a_{j'} + \tilde{m}_{j'} + H_j^t H_{t'}^s a_{t'}) , \\
F_{t'} &= F_{t'} g_{t'}
\end{aligned}$$

If $\phi \in \{a, m, g\}$ then we put

$$\phi(X, y) = \phi_{t'} X^t X^t$$

Now we can show that

$$\omega_{t'}^a \omega_{t'}^a = 0 , \quad \omega_{t'}^A \omega_{t'}^A = 0$$

$$\tilde{m}_{t'}^p u_{t'}^p = u_{t'}^p \quad \text{and} \quad a(u^A, u_a) = 0 . \quad (2.2)$$

From $F\tilde{m} = 0$ we have $F_{t'}^p u_{t'}^p = 0$ and hence, $H_{t'}^p u_{t'}^p = 0$. As $\tilde{m}(U_A, u_a) = 0$ by (2.1), we get $g(u_A, u_a) = 0$. This gives us that $\tilde{L}$ and $\tilde{M}$ are orthogonal with respect to g and a . From $F\tilde{m} = \tilde{m}F = 0$ we have

$$F_j^t \tilde{m}_t^s = 0, \quad F_t^t \omega_t^A = 0, \quad H_t^t \omega_t^A = 0, \quad (2.3)$$

$$\tilde{m}_j^p \tilde{m}_t^q a_{pq} = \tilde{m}_{jt} \quad (2.4)$$

By virtue of (1.2), we have

$$H_j^t H_t^s = \delta_s^j - \tilde{m}_j^s \quad (2.5)$$

From (2.4), (2.5) and $F_j^t \tilde{m}_t^s = 0$.

$$F_t^t \omega_t^A = 0, \quad H_t^t \omega_t^A = 0, \quad \text{we get}$$

$$H_j^t H_t^s g_{ts} + \tilde{m}_{jt} = g_{jt}, \quad \text{we obtain} \quad (2.6)$$

$$H_j^t H_t^s + \tilde{m}_j^s = \delta_j^s$$

Let $H_t^s g_{st} = H_{tt}$, then we get

$$H_j^t H_{tt} + \tilde{m}_{jt} = g_{jt} \quad (2.7)$$

From (2.6) and (2.7) we get

$$H_j^t H_{tt} = H_j^t H_t^s g_{ts}$$

or

$$H_j^t (H_{tt} - H_{tt}) = 0$$

which shows that H is symmetric.

3. CONFORMAL DIFFEOMORPHISM OF $F(K, -(-)^{K+1})$ MANIFOLD.

Let M^n be a C^∞ differentiable manifold. $\mathfrak{F}(M^n)$ be the ring of real valued differentiable function on M^n and $\mathfrak{X}(M^n)$ be the moduli of derivatives of $\mathfrak{F}(M^n)$. Then $\mathfrak{X}(M^n)$ is a Lie algebra over the real numbers and the elements of $\mathfrak{X}(M^n)$ are called vector fields.

Let (M^n, g) and $(\hat{M}^n, \hat{g})$ be two Riemannian manifolds and $\Psi : M^n \rightarrow \hat{M}^n$ be diffeomorphism. Let $X \in \mathfrak{X}(M^n)$, $\hat{X} \in \mathfrak{X}(\hat{M}^n)$ be the vector fields on M^n and $\hat{M}^n$ respectively. $\hat{X}$ corresponds to the X induced by Ψ . Then diffeomorphism Ψ is called conformal diffeomorphism provided there exists

$\rho \in \mathfrak{F}(M^n)$ such that

$$g^\circ(X^\circ, Y^\circ) * \Psi = e^{2\rho} g(X, Y) \quad \text{for all } X, Y \in \mathfrak{X}(M^n). \quad (3.1)$$

for $\sigma \in \mathfrak{F}(M^n)$ defined $\text{grad } \sigma \in \mathfrak{X}(M^n)$ by

$$g(\text{grad } \sigma, X) = X(\sigma) \quad \text{for all } X \in \mathfrak{X}(M^n) \quad (3.2)$$

In addition to (3.1) and (3.2) if

$$\Psi : M^n \rightarrow \mathring{M}^n, \quad \text{preserves } F(K, -(-)^{K+1}) \text{ structure i.e.}$$

$$F^\circ X^\circ = (FX)^\circ \quad (3.3)$$

where F and F° are (1,1) tensor fields with respect to M^n and $\mathring{M}^n$. If g° be the Riemannian metric in $\mathring{M}^n$, its metric satisfies the following

$$g^\circ(F^\circ X^\circ, F^\circ Y^\circ) = g^\circ(X^\circ, Y^\circ), \quad (3.4)$$

for all X°, Y° in $\mathring{L}^\circ$ that is g° restricted to $\mathring{L}^\circ$ is an almost product structure with respect to F° . The Nijenhuis tensor $N(X, Y)$ of F in M^n is expressed as follows, for all $X, Y \in \mathfrak{X}(M^n)$

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y] \quad (3.5)$$

We have [3]

$$[X^\circ, Y^\circ] = \{[X, Y]\}^\circ \quad (3.6)$$

By means of (3.3), (3.6) we get

$$N^\circ(X^\circ, Y^\circ) = \{N(X, Y)\}^\circ \quad \text{for all } X, Y \in \mathfrak{X}(M^n), \quad (3.7)$$

where N° is the Nijenhuis tensor corresponding to F° in $\mathring{M}^n$.

Since $\mathring{M}^n$ is also an $F(K, -(-)^{K+1})$ structure manifold therefore we can define complementary distribution corresponding to the projection operators $\tilde{I}$ and $\tilde{m}$. Let $\tilde{I}^\circ$ and $\tilde{m}^\circ$ be the projection operators in $\mathring{M}^n$ corresponding to the structure $F(K, -(-)^{K+1})$ which is defined as follows:

$$\tilde{I}^\circ \stackrel{\text{def}}{=} ((-)^{K+1} F^{K-1})^\circ, \quad \tilde{m}^\circ \stackrel{\text{def}}{=} (I - (-)^{K+1} F^{K-1})^\circ$$

or,

$$\tilde{I}^\circ \stackrel{\text{def}}{=} (-)^{K+1} F^{(K-1)^\circ},$$

$$\tilde{m}^\circ \stackrel{\text{def}}{=} I^\circ - (-)^{K+1} F^{(K-1)^\circ}$$

where I° is the identity operator in $\mathring{M}^n$. Now from (1.2), (3.3) and (3.8), it follows that in $F(K, -(-)^{K+1})$ structure manifold, we have:

$$\begin{aligned} \tilde{I}^\circ X^\circ &= (1)^{K+1} F^{(K-1)^\circ} X^\circ \\ &= ((-)^{K+1} F^{K-1} X)^\circ \\ &= (\tilde{I} X)^\circ. \end{aligned} \quad (3.9)$$

Similarly,

$$\begin{aligned} \tilde{m}^\circ X^\circ &= X^\circ - (-)^{K+1} F^{(K-1)^\circ} X^\circ \\ &= (X - (-)^{K+1} F^{K-1} X)^\circ \\ &= (\tilde{m} X)^\circ \end{aligned}$$

which shows that $\tilde{L}, \tilde{M}$ preserves the structure

THEOREM (3.1). If $\tilde{L}$ and $\tilde{M}$ be the distributions corresponding to the projection operators $\tilde{L}$ and $\tilde{M}$ in $\tilde{M}^n$ then we have

$$N(X, Y) = \{N(\tilde{L}X, \tilde{L}Y) + N(\tilde{L}X, \tilde{M}Y) + N(\tilde{M}X, \tilde{L}Y) + N(\tilde{M}X, \tilde{M}Y)\} \quad (3.10)$$

$$\begin{aligned} N(X, Y) = \{ & \tilde{L}N(\tilde{L}X, \tilde{L}Y) + N(\tilde{L}X, \tilde{M}Y) + \tilde{L}N(\tilde{M}X, \tilde{M}Y) + \tilde{M}N(\tilde{L}X, \tilde{L}Y) \\ & + N(\tilde{M}X, \tilde{L}Y) + \tilde{M}N(\tilde{M}X, \tilde{M}Y)\} \end{aligned} \quad (3.11)$$

PROOF. We have in consequence of (3.10)

$$N(\tilde{L}X, \tilde{L}Y) = [F\tilde{L}X, F\tilde{L}Y] - F[F\tilde{L}X, \tilde{L}Y] - F[\tilde{L}X, F\tilde{L}Y] + F^2[\tilde{L}X, \tilde{L}Y] \quad (3.12)$$

$$N(\tilde{L}X, \tilde{M}Y) = [F\tilde{L}X, F\tilde{M}Y] - F[F\tilde{L}X, \tilde{M}Y] - F[\tilde{L}X, F\tilde{M}Y] + F^2[\tilde{L}X, \tilde{M}Y] \quad (3.13)$$

$$N(\tilde{M}X, \tilde{L}Y) = [F\tilde{M}X, F\tilde{L}Y] - F[F\tilde{M}X, \tilde{L}Y] - F[\tilde{M}X, F\tilde{L}Y] + F^2[\tilde{M}X, \tilde{L}Y] \quad (3.14)$$

$$N(\tilde{M}X, \tilde{M}Y) = [F\tilde{M}X, F\tilde{M}Y] - F[F\tilde{M}X, \tilde{M}Y] - F[\tilde{M}X, F\tilde{M}Y] + F^2[\tilde{M}X, \tilde{M}Y] \quad (3.15)$$

Adding (3.12), (3.13), (3.14) and (3.15) we get

$$N(\tilde{L}X, \tilde{L}Y) + N(\tilde{L}X, \tilde{M}Y) + N(\tilde{M}X, \tilde{L}Y) + N(\tilde{M}X, \tilde{M}Y) = N(X, Y) \quad (3.16)$$

So in consequence of (3.7) we get

$$\begin{aligned} N^*(X^*, Y^*) &= \{N(\tilde{L}X, \tilde{L}Y) + N(\tilde{L}X, \tilde{M}Y) + N(\tilde{M}X, \tilde{L}Y) \\ &+ N(\tilde{M}X, \tilde{M}Y)\}^* = \{N(X, Y)\}^* \end{aligned}$$

This proves the first part of the theorem. The proof of the second part follows from (1.2)

4. INTEGRABILITY CONDITIONS OF $F(K, -(-)^{K+1})$ STRUCTURE

If the distribution $\tilde{L}$ in M^n is integrable then $N(\tilde{L}X, \tilde{L}Y)$ is exactly the Nijenhuis tensor of $F^* = \frac{F}{L}$

THEOREM (4.1). For any two vector fields X and Y we have

- (i) the distribution $\tilde{L}$ is integrable in M^n iff the distribution $\tilde{L}^\circ$ is integrable in $\tilde{M}^n$
- (ii) the distribution $\tilde{M}$ is integrable in M^n iff the distribution $\tilde{M}^\circ$ is integrable in $\tilde{M}^n$

PROOF. We know that the distribution $\tilde{L}$ is integrable in M^n iff $\tilde{M}[\tilde{L}X, \tilde{L}Y] = 0$ and the distribution $\tilde{M}$ is integrable in M^n iff $\tilde{L}[\tilde{M}X, \tilde{M}Y] = 0$, for any two vector fields $X, Y \in \mathfrak{X}(M^n)$. Hence in view of (3.6) and (3.7) and by means of integrability conditions of $\tilde{L}$ and $\tilde{M}$ [4] we obtain the proof of the theorem (4.1) (i) and (ii).

THEOREM (4.2). The distribution $\tilde{L}$ and $\tilde{M}$ are both integrable in M^n iff $\tilde{L}^\circ$ and $\tilde{M}^\circ$ are integrable in $\tilde{M}^n$

PROOF. The proof follows directly with the help of (4.1) (i) and (ii) and (3.10)

THEOREM (4.3). If the distribution $\tilde{L}$ is integrable in M^n then the almost product structure defined by $F^* \stackrel{\text{def}}{=} \frac{F}{L}$ on each integral manifold of $\tilde{L}$ is integrable in M^n iff the almost product structure defined by $\tilde{F}^* \stackrel{\text{def}}{=} \frac{F^\circ}{L^*}$ on each integral manifold of $\tilde{L}^\circ$ is integrable in $\tilde{M}^n$ provided $\tilde{L}^\circ$ is integrable in $\tilde{M}^n$.

PROOF. We suppose that the distribution $\tilde{L}$ is integrable in M^n then F induces on each integral manifold of $\tilde{L}$ an almost product structure if F is $F(K, -(-)^{K+1})$ structure. In both the cases the

structure is integrable iff the Nijenhuis tensor of M^n vanishes i.e., $N(\tilde{I}X, \tilde{I}Y) = 0$, or equivalently $\tilde{I}N(\tilde{I}X, \tilde{I}Y) = 0$ for any two vector fields X and Y

In view of (3.10) and $\tilde{I}\tilde{m} = \tilde{m}\tilde{I} = 0$ we get

$$N^\circ(\tilde{I}X^\circ, \tilde{I}Y^\circ) = \{N(\tilde{I}X, \tilde{I}Y)\}^\circ.$$

DEFINITION (4.1). We say that an $F(K, -(-)^{K+1})$ structure in M^n endowed with (1.1) tensor field F satisfying $F^K - (-)^{K+1}F = 0$ is p -partially integrable and the almost product structure $F^* \stackrel{\text{def}}{=} \frac{F}{L}$ is integrable

THEOREM (4.4). The $F(K, -(-)^{K+1})$ structure is p -partially integrable in M^n iff it is also p -partially integrable in $\tilde{M}^n$

PROOF. The proof follows in view of Def (4.1), Theorems (4.1) (i) and (4.3)

DEFINITION (4.2). We say that $F(K, -(-)^{K+1})$ structure to be partially integrable iff it is p -partially integrable and the distribution of $\tilde{M}$ is integrable

THEOREM (4.5). The structure $F(K, -(-)^{K+1})$ is partially integrable in M^n iff it is so in $\tilde{M}^n$

PROOF. The proof of the theorem follows from Definition (4.2) and Theorems (4.4) and (4.1) (i).

REFERENCES

- [1] YANO, K. and ISHIHARA, S., Integrability conditions of a structure satisfying $f^3 + f = 0$, *Quarterly J. of Maths.*, **15** (1964), pp. 217-22.
- [2] DAS, LOVEJOY S., Complete lift of a structure satisfying $F^K - (-)^{K+1}F = 0$, *International Journal of Mathematics and Mathematical Sciences*, **15**, 4 (1992), 803-808.
- [3] HICKS, N. J., Notes on Differential Geometry, D. Van Nostrand Company, Inc., Princeton, New York (1969).
- [4] GRAY, ALFRED, Some examples of almost Hermitian manifolds, *Illinois Jour. of Math*, **10** (1966), 353-366.
- [5] FLORENCE, GOULI-ANDREOU, On a structure defined by a tensor field F of type (1.1) satisfying $f^5 - f = 0$, *Tensor N.S.*, **36** (1982), 180-184