ON MATRIX CONVEXITY OF THE MOORE-PENROSE INVERSE

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ABSTRACT. Matrix convexity of the Moore-Penrose inverse was considered in the recent literature. Here we give some converse inequalities as well as further generalizations.

KEY WORDS AND PHRASES: Matrix convexity, generalized inverse


1. INTRODUCTION

Let $A$ and $B$ be two complex Hermitian positive definite matrices, and let $0 \leq \lambda \leq 1$. Then

$$[\lambda A + (1 - \lambda)B]^{-1} \leq \lambda A^{-1} + (1 - \lambda)B^{-1}$$

(1.1)

where $A \geq B$ means that $A - B$ is a positive semi-definite matrix.

This result, i.e., matrix convexity of the inverse function is an old result that appears explicitly in the papers [1,2,3,4,5] (see also the books [6, pp. 554-555] and [7, pp. 469-471]).

The related matrix convexity of the Moore-Penrose (generalized) inverse, denoted by $A^+$, was considered in paper [8,9,10]. The following was given in [10]:

Let $A$ and $B$ be two complex Hermitian positive semi-definite matrices of the same order. The inequality

$$[\lambda A + (1 - \lambda)B]^{+} \leq \lambda A^{+} + (1 - \lambda)B^{+}$$

(1.2)

for every $0 \leq \lambda \leq 1$ holds if and only if

$$R(A) = R(B)$$

(1.3)

where $R(A)$ is the range of $A$.

Two converses of (1.1) were obtained in [11]:

If $A$ and $B$ are complex Hermitian positive definite matrices and $0 \leq \lambda \leq 1$ is a real number, then

$$[\lambda A + (1 - \lambda)B]^{-1} \geq K(\lambda A^{-1} + (1 - \lambda)B^{-1})$$

(1.4)

and

$$[\lambda A + (1 - \lambda)B]^{-1} - (\lambda A^{-1} + (1 - \lambda)B^{-1}) \geq K A^{-1}$$

(1.5)

where

$$K = 4 \min \frac{\mu_i}{(1 + \mu_i)^2}, \quad \bar{K} = \min \frac{(\sqrt{\mu_i} - 1)^2}{-\mu_i},$$

(1.6a,b)

and the $\mu_i$ are the solutions of the equation

$$\det(B - \mu A) = 0.$$ 

(1.7)

In this note, we give analogous converses for (1.2), as well as some related results.

2. CONVERSES OF THE MATRIX CONVEXITY INEQUALITY OF THE MOORE-PENROSE INVERSE

Let $A$ and $B$ be two complex Hermitian positive semi-definite matrices of the same order such that (1.3) holds. Let $P$ be a unitary matrix such that $A = P \text{diag}(A_1, 0)P^*$ where $A_1$ is a diagonal positive definite matrix. When (1.3) holds, we have $B = P \text{diag}(B_1, 0)P^*$ where $B_1$ is positive definite.
**THEOREM 1.** Let $A$ and $B$ be two complex Hermitian positive semi-definite matrices of the same order such that (1.3) holds and let $0 \leq \lambda \leq 1$ then
\[
[\lambda A + (1 - \lambda)B]^* \geq K(\lambda A^* + (1 - \lambda)B^*)
\]  
where $K$ is defined by (1.6a) and the $\mu_i$ are the positive solutions of the equation
\[
det(B_1 - \lambda A_1) = 0.
\]

**THEOREM 2.** Let $A$, $B$ be defined as in Theorem 1. Then
\[
[\lambda A + (1 - \lambda)B]^* - (\lambda A^* + (1 - \lambda)B^*) \geq \bar{K}_1
\]
where $\bar{K}$ is defined by (1.6b) and the $\mu_i$ are positive solutions of the equation (2.2)

**PROOF.** By (1.4) and (1.5) we have
\[
[\lambda A_1 + (1 - \lambda)B_1]^{-1} \geq K(\lambda A_1^{-1} + (1 - \lambda)B_1^{-1})
\]
and
\[
[\lambda A_1 + (1 - \lambda)B_1]^{-1} - (\lambda A_1^{-1} + (1 - \lambda)B_1^{-1}) \geq \bar{K}_1
\]
where $K$ is defined by (1.6a), $\bar{K}$ by (1.6b) and the $\mu_i$ are solutions of (2.2). Since $PA^*P^* = (PAP^*)^+$, (2.1) follows from (2.4) and (2.3) from (2.5).

3. SOME RELATED RESULTS

Let $(Y, B, \mu)$ be a probability space and $A_y, y \in Y$ a collection of positive semi-definite matrices of the same order. Let $A_y = (a_{ijy}), 1 \leq i, j \leq n$ and $y \in Y$. Assume that $a_{ijy}$ as a function of $y$ is measurable for every $1 \leq i, j \leq n$. The following results were proved in [9,10].

Suppose there exists a set $D \in B$ such that $\mu(D) = 1$ and $A_{y_1}A_{y_2} = A_{y_2}A_{y_1}$ for every $y_1, y_2 \in D$. Let $R(A_y)$ be the same for all $y \in D \in B$. Suppose $A_y$ and $A_y^+$ as functions of $y$ are integrable with respect to $\mu$. Then
\[
\left[\int_Y A_y \mu(dy)\right]^+ \leq \int_Y A_y^+ \mu(dy).
\]

By $\int_Y A_y \mu(dy)$ we mean the matrix whose $(i, j)^{th}$ element is $\int_Y a_{ijy} \mu(dy)$.

**THEOREM 3.** If also all positive eigenvalues of $A_y$ for all $y \in Y$ are in the interval $[m, M]$ where
\[
0 < m < M,
\]
then the following inequalities hold:
\[
\int_Y A_y^+ \mu(dy) \leq \frac{(M + m)^2}{4mm} \left[\int_Y A_y \mu(dy)\right]^+
\]
and
\[
\int_Y A_y^+ \mu(dy) - \left[\int_Y A_y \mu(dy)\right]^+ \leq \frac{\sqrt{M} - \sqrt{m}}{Mm} I.
\]

**PROOF.** As in [9], we have that there exists an orthogonal matrix $C$ such that
\[
C^T A_y C = \text{diag}\{\lambda_{y_1}, \lambda_{y_2}, \ldots, \lambda_{ny}\}, \quad y \in Y
\]
where $\lambda_{y_1}, \lambda_{y_2}, \ldots, \lambda_{ny}$ are the eigenvalues of $A_y$. Since $A_y$ is positive semi-definite, each $\lambda_{y} \geq 0$. Let $k$ be the rank of $A_y$. We can assume without loss of generality that
\[
\lambda_{y_1}, \lambda_{y_2}, \ldots, \lambda_{ky} \neq 0 \quad \text{for every } y \in Y, \quad \text{and} \quad \lambda_{k+1,y} = \lambda_{k+2,y} = \ldots = \lambda_{ny} = 0 \quad \text{for every } y \in Y.
\]

Note that
\[
A_y^+ = C \text{diag}\left\{\frac{1}{\lambda_{y_1}}, \frac{1}{\lambda_{y_2}}, \ldots, \frac{1}{\lambda_{ky}}, 0, \ldots, 0\right\}C^T
\]
so that
Thus, we have
\[
K \left[ \int_Y A_y \mu(dy) \right]^{-1} - \int_Y \lambda_y \mu(dy) = C \text{ diag} \left\{ K \left( \int_Y \lambda_y \mu(dy) \right)^{-1} - \int_Y \lambda_y \mu(dy), 0, \ldots, 0 \right\} C^T
\]
where \( K = (M + m)^2 / (4Mm) \). The inequality
\[
K \left[ \int_Y \lambda_y \mu(dy) \right]^{-1} \int_Y \lambda_y \mu(dy)
\]
is the well-known Kantorovich inequality. Hence each diagonal element in the above diagonal matrix is nonnegative. This completes the proof of (3.2).

Similarly,
\[
\int_Y A_y \mu(dy) - \left[ \int_Y A_y \mu(dy) \right]^{-1} - K I = C \text{ diag} \left\{ \int_Y \lambda_y^{-1} \mu(dy) - \left( \int_Y \lambda_y \mu(dy) \right)^{-1} - K, \ldots, \int Y \lambda_y \mu(dy) \right\} C^T
\]
where \( K = \left( \frac{\sqrt{M} - \sqrt{m}}{Mm} \right)^2 \). The inequality
\[
\int_Y \lambda_y^{-1} \mu(dy) - \int_Y \lambda_y \mu(dy)^{-1} \leq K
\]
is a simple consequence of the following Mond-Shisha inequality \[12\]
\[
\int f - \left( \int f^{-1} \right)^{-1} \leq \left( \sqrt{M} - \sqrt{m} \right)^2
\]
where \( m \leq f \leq M, 0 < m < M \). Namely
\[
\frac{1}{M} \leq f \leq \frac{1}{m}
\]
so that by substituting \( f \rightarrow \frac{1}{f} \), we get
\[
\int f^{-1} - \left( \int f \right)^{-1} \leq \frac{\left( \sqrt{M} - \sqrt{m} \right)^2}{Mm} = K.
\]
Thus each diagonal element in the above diagonal matrix is non-positive. This completes the proof.

Moreover, we can consider the powers of \( A \) and \( A^+ \). For simplicity of notation, if \( r < 0 \), we shall use \( A^{(r)} \) for \( (A^+)^{-r} \). Note that \( (A^+)^{-r} = (A^{-r})^+ \).

**THEOREM 4.** Let \( R(A_y) \) be the same for all \( y \in D \in B \). Suppose \( A_y^+ \) and \( A_y^{(r)}, (r < 0 < s) \) as functions of \( y \) are integrable with respect to \( \mu \) Then
\[
\left[ \int_Y A_y^+ \mu(dy) \right]^s \geq \left[ \int_Y A_y^+ \mu(dy) \right]^{(r)}
\]

**PROOF.** As in the proof of (3.2) and (3.3), we have
\[
\left[ \int Y A_y^+ \mu(dy) \right]^s - \left[ \int Y A_y^+ \mu(dy) \right]^{(r)} = C \text{ diag} \left\{ \left( \int Y \lambda_y^+ \mu(dy) \right)^s - \left( \int Y \lambda_y^+ \mu(dy) \right)^r, \ldots, \left( \int Y \lambda_y^+ \mu(dy) \right)^s - \left( \int Y \lambda_y^+ \mu(dy) \right)^r \right\} C^T.
\]
Each diagonal element in the above diagonal matrix is nonnegative. This follows from the fact that if \( f^s \) and \( f^r \) are positive and integrable, the well-known inequality for means of orders \( s \) and \( r \) states that

\[
\left( \int f^r \right)^{1/r} \leq \left( \int f^s \right)^{1/s} \quad (r < 0 < s)
\]

which is the same as

\[
\left( \int f^s \right)^{r} \leq \left( \int f^r \right)^{s}.
\]

Similar consequences of converse inequalities for (3.5) (see [12] and [13], respectively) are the next two theorems

**Theorem 5.** Let the conditions of Theorem 4 be satisfied and let all positive eigenvalues of \( A_y \) for all \( y \in Y \) belong to the interval \([m, M]\) \((0 < m < M)\). Then the following inequality holds

\[
\left[ \int_Y A^s_y \mu(dy) \right]^{(r)} \geq \Delta \left[ \int_Y A^{(r)}_y \mu(dy) \right]^s
\]

where

\[
\Delta = \left\{ \frac{r(\gamma^s - \gamma^r)}{(s-r)(\gamma^r - 1)} \right\}^s \left\{ \frac{s(\gamma^r - \gamma^s)}{(r-s)(\gamma^s - 1)} \right\}^{-s}, \quad \gamma = M/m.
\]

**Theorem 6.** Let the conditions of Theorem 5 be satisfied. Then

\[
\left[ \int_Y A^s_y \mu(dy) \right]^{(r)} - \left[ \int_Y A^{(r)}_y \mu(dy) \right]^{(r)} \leq \Lambda I
\]

where

\[
\Lambda = \max_{\theta \in [0, 1]} \{ [\theta M^r + (1 - \theta)m^r]^s - [\theta M^s + (1 - \theta)m^s]^r] \}.
\]

Of course (3.2) and (3.3) are the special cases \( r = -1, s = 1 \) of (3.6) and (3.8).

**References**


