ABSTRACT. In the present note, we define operator spaces with \( n \)-hyper-reflexive property, and prove \( n \)-hyper-reflexivity of some operator spaces.

KEY WORDS AND PHRASES. Operator algebras on Hilbert spaces, reflexivity, hyper-reflexivity

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1. INTRODUCTION

Let \( H \) be a Hilbert space, and \( B(H) \) be the algebra of all bounded linear operators on \( H \). It is well known that \( B(H) \) is the dual space of the Banach space of trace class operators. If \( T \in B(H) \), \( R \subset B(H) \), and \( n \) is a positive integer, then \( H^{(n)} \) denotes the direct sum of \( n \) copies of \( H \), \( T^{(n)} \) denotes the direct sum of \( n \) copies of \( T \) acting on \( H^{(n)} \) and \( R^{(n)} = \{ T^{(n)} | T \in R \} \). Let \( P(H) \) be the set of all orthogonal projections in \( B(H) \). For any subspace \( R \subset B(H) \), we will denote by \( l(R) \) the collection of all maximal elements of the set

\[ \{(Q, P) | (Q, P) \in P(H) \times P(H), QRT = 0\} \]

with respect to the natural order. It can be seen that if \( R \) is a unital subalgebra of \( B(H) \), then

\[ l(R) = \{1 - P, P) | P \in \text{lat } R\} \]

where \( \text{lat } R \) is lattice of all invariant subspace of \( R \). Recall that an algebra \( R \subset B(H) \) is transitive if \( \text{lat } R \{0, 1\} \), and reflexive if the only operators that leave invariant all of the invariant subspaces of \( R \) are the operators belonging to \( R \). Generalizing this notion, we say that an operator space \( R \subset B(H) \) is transitive if \( l(R) = \{(0, 1), (1, 0)\} \) (this is equivalent to \( \overline{R}x = H \) for any \( x \in H - \{0\} \)), and is reflexive if

\[ R = \{T \in B(H) | QTP = 0 \text{ for every } (Q, P) \in l(R)\} . \]

In other words, \( R \) is reflexive if the seminorms \( d(T, R) \) and \( \sup \{\|QTP\| | (Q, P) \in P(R)\} \) vanish on \( R \) simultaneously, where \( d(T, R) \) is the distance from \( T \) to \( R \). It can be seen that

\[ d(T, R) \geq \sup \{\|QTP\| | (Q, P) \in l(R)\} \]

for any \( T \in B(H) \).

Reflexive operator space \( R \subset B(H) \) is called hyper-reflexive if there exists some constant \( C \geq 1 \) such that

\[ d(T, R) \leq C \sup \{\|QTP\| | (Q, P) \in l(R)\} \]

for any \( T \in B(H) \), (see \([1-5]\))

In [4], an example of non hyper-reflexive operator algebras is constructed.
In the present note, we define operator spaces with n-hyper-reflexive property, and prove n-hyper-reflexivity of some operator spaces.

The operator space $R \subset B(H)$ is called n-reflexive if $R^{(n)}$ is reflexive. It can be shown that

$$d(T, R) \geq \sup \{\|QT^{(n)}P\| | (Q, P) \in l(R^{(n)})\}$$

for any $T \in B(H)$ and $n \in N$.

We say that the n-reflexive operator space $R \subset B(H)$ is n-hyper-reflexive if there exists some constant $C \geq 1$ such that

$$d(T, R) \leq C \sup \{\|QT^{(n)}P\| | (Q, P) \in l(R^{(n)})\}$$

for any $T \in B(H)$.

It is easily seen that if $R$ is n-reflexive (n-hyper-reflexive) then it is $k$-reflexive ($k$-hyper-reflexive) for every $k > n$.

2. MAIN RESULT

Let us consider in $B(H)$ the following operator equation

$$\sum_{i=1}^{n} A_iXB_i = X.$$ (2.1)

The space of all solutions of the equation (2.1) will be denoted by $R$.

**Proposition 1.** $R$ is $(n + 1)$-reflexive.

**Proof.** For given any $x, y \in H - \{0\}$, put

$$z = (B_1x, ..., B_nx, x) \in H^{(n+1)} \quad \text{and} \quad y = (A^*_1y, ..., A^*_ny, -y) \in H^{(n+1)}.$$

Let $P_x$ and $Q_y$ be the one-dimensional projections on one-dimensional subspaces $\{C_x\}$ and $\{C_y\}$ respectively. From (2.1), we have $(Q_y, P_x) \in l(R^{(n+1)})$. On the other hand, it is easy to see that any $T \in B(H)$ is a solution of equation (2.1) if and only if $Q_y T^{(n+1)} P_x = 0$. This completes the proof.

We will assume that, in case $n > 1$, the coefficients of equation (2.1) satisfy the following conditions

$$\|A_i\| \leq 1, \quad \|B_i\| \leq 1, \quad A_iA_j = B_iB_j = 0 \quad (1 \leq i < j \leq n). \quad (2.2)$$

The purpose of this note is to prove the following.

**Theorem 2.** The space $R$ of all solutions of (2.1) and (2.2) is $(n + 1)$-hyper-reflexive.

To prove Theorem 2 we need some preliminary results.

Let $Y$ be a Banach space with $Y^* = X$ and $S$ be a weak* continuous linear operator on $X$ with uniformly bounded degree, $\|S^n\| \leq C(n \in N)$. Denote by $E$ the space of all fixed points of $S$, $E = \{x \in X | Sx = x\}$. If $x_0 \in E$, then for any $x \in X$ we have

$$\|S^n x - x\| = \|S^n(x - x_0) - (x - x_0)\| \leq (C + 1)\|x - x_0\|$$

and consequently

$$d(x, E) \geq \frac{1}{C + 1} \sup_n \|S^n x - x\|$$

**Proposition 3.** Under the above assumptions,

$$d(x, E) \leq \sup_n \|S^n x - x\|$$

for any $x \in X$.

**Proof.** Since $E$ is a weak* closed subspace of $X$, there exists a subspace $M \subset Y$ such that $M^\perp = E$, where $M^\perp$ is the annihilator of $M$. It can be seen that the set $\{Ty - y | y \in Y\}$ weak* generates $M$, where $T$ is the preadjoint of $S$, that is, $T^* = S$. Let $x \in X$ and let $K(x)$ be the weak* closure of the convex hull of the set $\{S^n x | n \in N\}$. By Alaoglu's theorem, $K(x)$ is weak* compact. We will show that $K(x) \cap E \neq \emptyset$ for any $x \in X$. Suppose that $K(x) \cap E = \emptyset$. By Hahn-Banach separating theorem, there exists $y_0 \in M$ such that
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\[ \inf_{a \in K(x)} |\langle a, y_0 \rangle| = \sigma > 0 \]

where \( \langle , \rangle \) is the duality between \( X \) and \( Y \).

Put

\[ x_n = \frac{1}{n} \sum_{k=1}^{n} S^k x. \]

Then \( x_n \in K(x) \) and \( \|x_n\| \leq C\|x\| \). Now, we will prove that

\[ \lim_{n \to \infty} |\langle x_n, y_0 \rangle| = 0. \quad (23) \]

Since \( (x_n) \) is a bounded set, it is sufficient to prove the equality \((23)\) in case \( y_0 = Ty - y, (y \in Y) \). In that case

\[ \langle x_n, Ty - y \rangle = \langle Sx_n - x_n, y \rangle = \frac{1}{n} \langle S^{n+1} x - Sx, y \rangle \to 0. \]

Now, suppose that \( \|S^n x - x\| \leq \delta \) for some \( \delta > 0 \) and any \( n \in \mathbb{N} \). It is easy to see that \( \|a - x\| \leq \delta \) for any \( a \in K(x) \). Let \( a_0 \in K(x) \cap E \). Then \( \|a_0 - x\| \leq \delta \) and consequently \( d(x, E) \leq \delta \).

**PROOF OF THEOREM 2.** For any \( A \in B(H) \) we denote by \( L_A \) and \( R_A \) the left and right multiplication operators \( L_A : X \to AX, R_A : X \to XA \) on \( B(H) \) respectively. Then we may write equation \((21)\) as

\[ \left( \sum_{i=1}^{n} L_A R_B \right) X = X. \]

Thus, the solution space \( R \) of \((21)\) coincide with the set of all fixed points of the operator

\[ S = \sum_{i=1}^{n} L_A R_B. \]

It is easily seen that \( S \) is a weak* continuous linear operator on \( B(H) \). Moreover, under assumption \((22)\), it can be shown (by induction) that

\[ S^k = \sum_{i=1}^{n} L_{A^i} R_{B^i}. \]

and consequently \( \|S^k\| \leq n \).

By Proposition 3, for any \( T \in B(H) \) we have

\[ d(T, R) \leq \sup_k \|S^k(T) - T\| = \sup_k \left\| \sum_{i=1}^{n} A_i^k T B_i^k - T \right\| \]

\[ = \sup_k \sup_{\|x\| \leq 1, \|y\| \leq 1} \left| \sum_{i=1}^{n} (TB_i^k x, A_i^k y) - (Tx, y) \right|. \]

For \( \|x\| \leq 1 \) and \( \|y\| \leq 1 \), let \( x_k = (B_1^k x, \ldots, B_n^k x, x), y_k = (A_1^k y, \ldots, A_n^k y, -y) \). It can be seen that

\[ (R^{(n+1)} x_k, y_k) = 0 \quad \text{and} \quad \|x_k\|^2 \leq n + 1, \|y_k\|^2 \leq n + 1 (k \in \mathbb{N}). \]

Therefore

\[ d(T, R) \leq (n + 1) \sup \left\{ \|T^{(n+1)} x, y\| \mid (R^{(n+1)} x, y) = 0, \|x\| = \|y\| = 1 \right\}. \]

Let \( P_x Q_y \) be the one-dimensional projections (as in the proof of Proposition 1). Then we obtain
This completes the proof

**COROLLARY 4.** Let $A, B \in B(H)$ with $\|A\| \leq 1, \|B\| \leq 1$ Then, the solution space $R$ of the equation

$$AXB = X$$

is 2-hyper-reflexive with constant $C = 2$

Generally speaking, the solution space of equation (2.4) may be reflexive For example, if $Q, P \in P(H),$ then the solution space of equation

$$QXP = X$$

is reflexive Hyper-reflexivity (with constant $C = 1$) of the solution space of equation (2.5) was proved in [3]

Note that the space of all Toeplitz operators $\tau$ coincide with the solution space of (2.4) in case $A = U^*$ and $B = U,$ where $U$ is a unilateral shift operator on Hardy space $H^2$ [6]

Consequently, $\tau$ is a 2-reflexive by Proposition 1 Using Theorem 2, we can deduce even more

**COROLLARY 5.** The space of all Toeplitz operators $\tau$ is 2-hyper-reflexive, with constant $C = 2$

In other words

$$d(T, \tau) \leq 2 \sup \left\{ \|QT^{(2)} P\| \left| (Q, P) \in l^2(\tau) \right. \right\}$$

for any $T \in B(H^2)$

On the other hand we have the following

**PROPOSITION 6.** The space of all Toeplitz operators $\tau$ is transitive (consequently $\tau$ is not reflexive)

**PROOF.** Suppose that $\tau$ is nontransitive Then there exists $f, g \in H^2 - \{0\}$ such that $(Tf, g) = 0$ for every $T \in \tau$ If we put in last equality $T = U^n$ and $T = U^{*n}$ $(n = 0, 1, 2, \ldots),$ then we obtain that the Fourier coefficients of the function $f, g$ are zero Since $f, g = 0$ $a.e.$, one of these functions vanishes $a.e.$ on some subset of the unit circle with positive Lebesque measure By F and M Riesz uniqueness theorem [6], one of these functions is zero

Hyper-reflexivity of algebras of analytic Toeplitz operators was proved in [5]

**REFERENCES**


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