ON THE DIAPHONY OF ONE CLASS OF ONE-DIMENSIONAL SEQUENCES

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ABSTRACT In the present paper, we consider a problem of distribution of sequences in the interval [0,1), the so-called 'Pr-sequences'. We obtain the best possible order $O(N^{-1} (\log N)^{1/2})$ for the diaphony of such Pr-sequences. For the symmetric sequences obtained by symmetrization of Pr-sequences, we get also the best possible order $O(N^{-1} (\log N)^{1/2})$ of the quadratic discrepancy.

KEY WORDS AND PHRASES Distribution of sequences, quadratic discrepancy and Pr-sequences

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INTRODUCTION

Let $\sigma = (x_n)_{n=0}^\infty$ be an infinite sequence in the unit interval $E = [0,1)$. For every real number $x \in E$ and every positive integer $N$ we denote $A_N(\sigma; x)$ the number of terms $x_n$, $0 \leq n \leq N-1$, which are less than $x$.

The sequence $\sigma$ is called uniformly distributed in $E$ if for every real number $x \in E$ we have

$$\lim_{N \to \infty} A_N(\sigma; x)N^{-1} = x.$$

The systematic study of the theory of uniformly distributed sequences was initiated by Weyl [1].

A classical measure for the irregularity of the distribution of a sequence $\sigma$ in $E$ is its quadratic discrepancy $T_N(\sigma)$, which is defined for every positive integer $N$ as

$$T_N(\sigma) = \left( \int_0^1 \left| A_N(\sigma; x)/N - x \right|^2 dx \right)^{1/2}.$$ 

The irregularity of distribution with respect to the quadratic discrepancy was first studied by Roth [2].

In 1976, Zinterhof (see [3,4]) proposed a new measure for distribution, which he named diaphony. The diaphony $F_N(\sigma)$ of $\sigma$ is defined for every positive integer $N$ as

$$F_N(\sigma) = \left( 2 \sum_{h=1}^{N-1} h^{-2} \left| S_N(\sigma; h) \right| ^2 \right)^{1/2}$$

where

$$S_N(\sigma; h) = \sum_{n=0}^{N-1} \exp(2\pi i h x_n)$$

signify trigonometric sum of $\sigma$.

We note that the diaphony of $\sigma$ can be written in the form

$$F_N(\sigma) = (N^{-2} \sum_{n,k=0}^{N-1} g(x_n - x_k))^{1/2}$$

where

$$g(x) = \frac{\pi^2}{2} \left( 2x^2 - 2x + 1/3 \right)$$

It is well known (see [5], p 115, [4]) that both equalities

$$\lim_{N \to \infty} T_N(\sigma) = 0$$

and

$$\lim_{N \to \infty} F_N(\sigma) = 0$$

are equivalent to the definition that the sequence $\sigma$ is uniformly distributed in $E$. 
Using the well-known theorem of Roth [2] it can be proved (see Neiderreiter [7], p 158; 
Proinov [8]) that for any infinite sequence $\sigma$ in $E$, the estimate
\[
T_N(\sigma) > 214^{-1}N^{-1}(\log N)^{1/2}
\]
holds for infinitely many integers $N$. The exactness of the order of magnitude of this estimate was 
proved by Proinov ([9], [10], [11]).

Proinov [8] proved that for any sequence $\sigma$ in $E$ the estimate
\[
F_N(\sigma) > 68^{-1}N^{-1}(\log N)^{1/2}
\]
holds for infinitely many $N$.

From (1.1) and (1.2) becomes clearly that the best possible order of diaphony and quadratic 
discrepancy of every sequence $\sigma$ in $E$ is $O(N^{-1}(\log N)^{1/2})$.

2. A SEQUENCE OF $r$-ADIC RATIONAL TYPE.

2.1 CONSTRUCTION OF SEQUENCE OF $r$-ADIC RATIONAL TYPE

In this part we generalize Sobol's ([12], [5], p 117, [13], p. 23) construction of sequences of binary 
rational type.

Let $r \geq 2$ is fixed integer. We consider the infinite matrix
\[
(v_{s,j}) = \begin{pmatrix}
v_{11} & v_{21} & \cdots & \cdots \\
v_{12} & v_{22} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
\]
where for every $s, j = 1, 2, \cdots, v_{s,j} \in \{0, 1, \cdots, r-1\}$. We suppose that in every column, the 
quantity of $v_{s,j}$, which are different from zero is a positive integer number, i.e., $v_{s,j} = 0$ for $j$ sufficiently 
big. Such matrix we shall call guiding matrix.

To every column of the matrix (2.1) corresponds a $r$-adic rational numbers
\[
V_j = 0, v_{s,1} v_{s,2} \cdots v_{s,j} \cdots (s = 1, 2, \cdots)
\]
The numbers determined in (2.2) are called guiding numbers.

We signify $N_0 = N \cup \{0\}$, with $N$ the set of natural integers.

A sequence of $r$-adic rational type (or RP-sequence) is a sequence $(\varphi(i))_{i=0}^{\infty}$, which is generated by 
the guiding matrix $(v_{s,j})$ in the following way: If in the $r$-adic number system
\[
i = e_m e_{m-1} \cdots e_1
\]
then in the $r$-adic number system
\[
\varphi(i) = 0, W_1^* W_2^* \cdots W_m^*
\]
where for $j = 1, 2, \cdots, m$
\[
W_j = e_j V_j = V_j^* V_j^{*} \cdots V_j^{*} \underbrace{e_j - \text{ terms}}_{e_{s+1} - \text{ terms}}
\]
and $*$ is the operation of the digit-by-digit addition modulo $r$ of elements of $Z_r = \{0, 1, \cdots, r-1\}$.

A RP-sequence $(\varphi(i))_{i=0}^{\infty}$, which is generated by the guiding matrix $(v_{s,j})$ can be also constructed 
by following the three mentioned below rules:

1. $\varphi(0) = 0$.
2. If $i = r^s (s \in N_0)$, then $\varphi(i) = V_{s+1}$.
3. If $r^s < i < r^{s+1}$, then $\varphi(i) = e_{s+1} \varphi(r^s) + (i - e_{s+1} r^s)$, where $e_{s+1}$ is higher significant digit 
in $r$-adic development of $i$ and $e_{s+1} \varphi(r^s) = \underbrace{V_{s+1}^* V_{s+1}^{*} \cdots V_{s+1}^{*}}_{e_{s+1} - \text{ terms}}$.

Obviously the operation $*$ has commutative and associative property.

We shall prove that the two definitions of the $PR$-sequences are equivalent.

Let us suppose that the first definition is valid for $RP$-sequence.

1. If $i = 0$, then obviously $\varphi(i) = 0$.
2. If $i = r^s (s \in N_0)$, then $\varphi(i) = V_{s+1}$.
Let us assume that \( r^s < i < r^{s+1} \) and \( i = (e_{s+1}, e_s, \cdots, e_1)_r \). Since the operator \( \cdot \) is commutative and associative we have
\[
\varphi(i) = 0, ((e_1 V_1)^s \cdots (e_s V_s))((e_{s+1} V_{s+1})^s).
\]
Since \( V_{s+1} = \varphi(r^s) \) and \( i - e_{s+1} r^s = (e_s, e_{s-1}, \cdots, e_1)_r \), then \( \varphi(i - e_{s+1} r^s) = 0, ((e_1 V_1)^s \cdots (e_s V_s)) \).

Finally \( \varphi(i) = e_{s+1} \varphi(r^s)^s \varphi(i - e_{s+1} r^s) \). The three rules in the second definition for RP-sequence are proved.

Reversely, let the second definition for PR-sequence is valid and \( i \) is given positive integer. Then there exists uniquely positive integer \( s \) that \( r^s < i < r^{s+1} \). We shall prove definition 1 by induction on \( s \).

If \( s = 0 \), then \( 1 \leq i < r \) and
\[
\varphi(i) = i \varphi(r^0)^s \varphi(0) = 0, i V_1.
\]
We make inductive supposition that for some \( s \in N \) and every integer \( i \), \( r^{s+1} < i < r^s \) definition 1 holds.

Let us assume that \( r^s < i < r^{s+1} \) and \( i = (e_{s+1} e_s, \cdots, e_1)_r \). From rule 3 we have
\[
\varphi(i) = e_{s+1} \varphi(r^s)^s \varphi(i - e_{s+1} r^s).
\]
If we denote \( j = i - e_{s+1} r^s \), then \( j = (e_s, e_{s-1}, \cdots, e_1)_r \) and \( r^{s+1} < j < r^s \). Then by inductive supposition
\[
\varphi(j) = 0, (e_1 V_1)^s \cdots (e_s V_s)^s.
\]
By rule 2, \( \varphi(r^s) = V_{s+1} \) and we have
\[
\varphi(i) = 0, (e_1 V_1)^s \cdots (e_s V_s)^s (e_{s+1} V_{s+1}).
\]
Definition 1 holds for every positive integer \( s \).

In the following lemma we give a property of the functions \( \varphi \).

**Lemma 2.1.** Let \( (u_{s,j}) \) is an arbitrary guiding matrix, and \( \varphi(i)^{+\infty}_{i=0} \) is RP-sequence, which is generated by \( (u_{s,j}) \). Let \( \nu, m, n \) be integer numbers such that \( \nu \in N_0 \), \( 0 \leq n < r^\nu \) and \( m \equiv 0 (mod r^\nu) \). Then we have
\[
\varphi(m) = \varphi(m)^s \varphi(n).
\]
The proof of the lemma is obvious.

For every integer \( a \in Z_r \) we define \( \overline{a} \) the only integer, which is a solution of the equation
\[
a + \overline{a} \equiv 0 (mod r).
\]
If \( \alpha = 0, \alpha_1, \alpha_2, \cdots, \alpha_t \), where, \( \alpha = 0, \alpha_1, \alpha_2, \cdots, \alpha_t \), \( t \in Z_r \), then we define
\[
\overline{\alpha} = 0, \overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_t.
\]

**2.2. SEQUENCES OF \( r \)-ADIC RATIONAL TYPE, WHICH ARE \( P_r \)-SEQUENCES.**

The theory of the \( P_r \)-sequences was first studied by Faure ([14],[15]) and generalized by Neiderreiter ([16],[17]).

A \( r \)-adic elementary interval is an interval
\[
l_{m,j} = ([j - 1]/r^m, j/r^m),
\]
in which \( 1 \leq j \leq r^m \), for any integer \( m \).

Let \( N = r^m \). We shall call the net
\[
X = (x_0, x_1, \cdots, x_{N-1})
\]
be a net of type \( P_r^m \) (or \( P_r^m \)-type), if every \( r \)-adic elementary interval \( l_{m,j} \), having length \( 1/N \) contain one point of the net \( X \).

A \( r \)-adic section of the sequence \( X = (x_i)_{i=0}^{\infty} \) is a set of terms \( x_i \), with numbers \( i \), satisfying the inequalities
\[
k r^s \leq i < (k + 1) r^s,
\]
for every integers \( k \) and \( s \), such that \( k = 0, 1, \cdots ; s = 1, 2, \cdots \).

The sequence \( (x_i)_{i=0}^{\infty} \) is called a sequence of type \( P_r \) (or \( P_r \)-sequence) if every \( r \)-adic section is a \( P_r^m \)-net.

**Theorem 2.1.** Let in the guiding matrix \( (u_{s,j}) \) every \( u_{s,s} = 1 \) and for \( j > s \) every \( u_{s,j} = 0 \), i.e.,
Then the corresponding RP-sequence is \( P_r \)-sequence

**PROOF.** We choose arbitrary \( r \)-adic section of the RP-sequence \( (\varphi(i))_{i=0}^{\infty} \), the length of which is \( r^m \). We write the numbers \( i \), belonging to this section in the \( r \)-adic number system:

\[
i = c_\mu c_{\mu-1} \cdots c_{m+1} e_m e_{m-1} \cdots e_1,
\]

where \( c_k \) are fixed and \( e_k \) are arbitrary \( r \)-adic numbers

We choose now an arbitrary \( r \)-adic interval \( l \), with length \( |l| = r^{-m} \). In the \( r \)-adic system this interval is determined by the inequality

\[
0, a_1 a_2 \cdots a_m \leq x < 0, a_1 a_2 \cdots a_m + 0, \frac{0}{m - \text{zeros}},
\]

where \( a_1, \cdots, a_m \) are \( r \)-adic numbers

We shall prove, that for every choice of the numbers \( c_k \) and \( a_k \) among the numbers \( i \), in the form (2.4) there exists exactly one \( i \), for which \( \varphi(i) \in l \).

In the \( r \)-adic number system we write

\[
\varphi(i) = 0, g_{i,1} g_{i,2} \cdots g_{i,j} \cdots.
\]

From (2.3) we have

\[
g_{i,j} = e_1 v_{i,j}^* \cdots e_m v_{m,j}^* c_{m+1} v_{m+1,j}^* \cdots c_\mu v_{\mu,j},
\]

where the sense of \( e_k v_{k,j} \) is the same as in (2.3).

The condition \( \varphi(i) \in l \) is equivalent to the following conditions

\[
g_{i,j} = a_j, \text{ for } 1 \leq j \leq m.
\]

We get that for each \( j \), \( 1 \leq j \leq m \)

\[
g_{i,j} = (e_1 v_{i,j}^* \cdots e_m v_{m,j}^*)^*(c_{m+1} v_{m+1,j}^* \cdots c_\mu v_{\mu,j}),
\]

from which we get

\[
e_1 v_{i,j}^* \cdots e_m v_{m,j} = a_j^*(c_{m+1} v_{m+1,j}^* \cdots c_\mu v_{\mu,j}) (1 \leq j \leq m) \tag{2.5}
\]

Let us call \( f_j \) the right-side of (2.5) for \( 1 \leq j \leq m \). Having in mind that for \( s = 1, 2, \cdots \), \( v_{k,s} = 1 \) and in case \( j > s \), \( v_{s,j} = 0 \), the system (2.5) become

\[
e_j^* v_{j,j}^* e_{j+1} v_{j+1,j}^* \cdots e_m v_{m,j} = f_j (1 \leq j \leq m).
\]

In this system the unknowns \( e_1, e_2, \cdots, e_m \) are successively so determined that it has only one solution.

The theorem is proved.

In the following lemma we shall show some property of \( P_r \)-sequences.

**LEMMA 2.2.** Let \( N = r^\nu \) where \( \nu \in N_0 \). For every guiding matrix \( (v_{x,j}) \) in which \( v_{x,s} = 1 \) and \( v_{x,j} = 0 \) for \( j > s (s = 1, 2, \cdots) \) and for the \( RP \)-sequence \( (\varphi(i))_{i=0}^{\infty} \), which is product of \( (v_{x,j}) \) we have

\[
\varphi(i) : 0 \leq i < r^\nu = \{ j/N : 0 \leq j < N \} \tag{2.6}
\]

**PROOF.** We shall make the proof by induction on \( \nu \). If \( \nu = 0 \) and \( \nu = 1 \), then we make directly examination.

We make inductive supposition, that for some \( \nu \in N \) the equality (2.6) is true and for \( j = 0, 1, \cdots, r - 1 \) we consider the multitudes \( A_j = \{ \varphi(i) : j r^\nu \leq i < (j + 1) r^\nu \} \). Then obviously

\[
A = \bigcup_{j=0}^{r-1} A_j \tag{2.7}
\]

where \( A = \{ \varphi(i) : 0 \leq i < r^{\nu+1} \} \).

We consider that \( j = 0 \). By the inductive supposition

\[
A_0 = \{ \varphi(i) : 0 \leq i < r^\nu \} = \{ m/r^{\nu+1} : 0 \leq m < r^{\nu+1}, m \equiv 0 \ (mod \ r) \} \tag{2.8}
\]
Let us now consider that \( 1 < j < r - 1 \). We shall prove the following equality
\[
A_j = \{ m/r^\nu + 1 : 0 \leq m < r^{\nu + 1}, m \equiv j \pmod{r} \}.
\] (2.9)

Let \( j, 1 \leq j \leq r - 1 \) be fixed integer and consider that \( j r^\nu \leq i < (j + 1) r^\nu \). Let us represent \( i \) in the form \( i = j r^\nu + k \), where \( 0 \leq k < r^\nu \).

Then by Lemma 2.1 we have
\[
\varphi(i) = \varphi(j r^\nu)^\nu \varphi(k) \quad \text{for } i = jr^\nu + k.
\] (2.10)

It is obvious that
\[
\varphi(jr^\nu) = \frac{V_{r^{\nu+1}} \cdots V_{r^\nu}}{j - \text{terms}}
\] (2.11)

Let us put
\[
\varphi(j r^\nu) = 0, w_{r^\nu+1,1} w_{r^\nu+1,2} \cdots w_{r^\nu+1,v+1}.
\]

From (2.11) is clear that \( w_{r^\nu+1,v+1} = j \). Let \( k \) has \( r \)-adic development \( k = k_v k_{v-1} \cdots k_1 \). Then
\[
\varphi(k) = 0, (k_1 V_1)^{\nu} \cdots (k_v V_v)^{\nu}
\]
\[
\varphi(k) = 0, a_1 a_2 \cdots a_v, \quad \text{where } a_s \in \{ 0, 1, \cdots, r - 1 \}
\]
(2.12)

From (2.10), (2.11) and (2.12) we get
\[
\varphi(i) = 0, (a_1 w_{r^\nu+1,1})^\nu \cdots (a_v w_{r^\nu+1,v})^\nu j = 0, b_1 b_2 \cdots b_v j
\] (2.13)

When \( 0 \leq k < r^\nu \), then \( 0 \leq (b_1 b_2 \cdots b_v) r < r^\nu \) and from (2.13) we get that for \( 1 \leq j \leq r - 1 \)
\[
A_j = \{ \varphi(i) : j r^\nu \leq i < (j + 1) r^\nu \} = \{ m/r^{\nu + 1} : 0 \leq m < r^{\nu + 1}; m \equiv j \pmod{r} \}
\]

The inequalities (2.9) are proved.

By induction on \( \nu \) the lemma is proved.

**Lemma 2.3** Let \( \{ i \}_0^\nu \) be a \( P \)-sequence. Then for every \( \nu \in \mathbb{N}_0 \) holds the equality
\[
\{ \varphi(m + j) : m \equiv 0 \pmod{r^\nu}, 0 \leq j < r^\nu \}
\]
\[
= \{ \varphi(m) + \varphi(j) \pmod{1} : m \equiv 0 \pmod{r^\nu}, 0 \leq j < r^\nu \}
\]
(2.14)

**Proof.** Let us consider that \( m = kr^\nu \), for some positive integer \( k \). The equality (2.14) is equivalent to the equality
\[
\{ \varphi(m + j) : m \equiv 0 \pmod{r^\nu}, 0 \leq j < r^\nu \}
\]
\[
= \bigcup_{l=0}^{r^\nu - 1} \{ \varphi(m + j) : m \equiv 0 \pmod{r^\nu}, lr \leq j < (l + 1)r \}
\] (2.15)

First, we shall prove that for every fixed \( l, 0 \leq l \leq r^\nu - 1 \) exists uniquely \( l' \leq l' < r^{\nu - 1} \), such that
\[
\{ \varphi(m + j) : m = kr^\nu, lr \leq j < (l + 1)r \}
\]
\[
= \{ \varphi(m) + \varphi(j) \pmod{1} : m = kr^\nu, l'r \leq j < (l' + 1)r \}
\] (2.16)

Let \( k = (k_n k_{n-1} \cdots k_1) \). Then we have
\[
\varphi(m) = g_1^m g_2^m \cdots g_n^m,
\]
where for \( 1 \leq i \leq n + \nu \) \( g_i^m = \sum_{h=1}^{n} k_h v_{h,i} \pmod{r} \).

Let \( 0 \leq l < r^{\nu - 1} \) be fixed integer and \( l = (l_{\nu-1} \cdots l_1) \). Then \( lr = (l_{\nu-1} \cdots l_1 \nu) \).

When \( j \) is fixed integer and \( lr \leq j < (l + 1)r \), we have \( j = (l_{\nu-1} \cdots l_1 l_0) \), where \( l_{\nu-1}, \cdots, l_1 \) are fixed integers and \( l_0 \) takes \( r \) different values in the set \( \{ 0, 1, \cdots, r - 1 \} \). Let \( \varphi(j) = 0, a_1 \cdots a_v \), where
\[
a_1 = l_0 + \sum_{h=1}^{\nu-1} e_h v_{h+1,1} \pmod{r}
\]
\[
a_2 = l_1 + \sum_{h=1}^{\nu-1} e_h v_{h+1,2} \pmod{r}
\]
\[
\vdots
\]
\[
a_{v-1} = l_{v-2} + l_{v-1} v_{v,v-1} \pmod{r}
\]
\[
a_v = 1_{v-1}.
\]

It is obvious that \( a_1 \) takes \( r \) different values in the set \( \{ 0, 1, \cdots, r - 1 \} \).

From the Lemma 2.1 we have
\[ \varphi(m + j) = (0, g_1^m g_2^m \cdots g_{n+\nu}^m) \ast (0, a_1 a_2 \cdots a_\nu) = 0, b_1 b_2 \cdots b_\nu g_{n+\nu}^m \cdots g_{n+\nu}^m \]

where

\[ b_\nu = g_\nu^m + a_\nu \pmod r \]
\[ b_{\nu-1} = g_{\nu-1}^m + a_{\nu-1} \pmod r \]
\[ b_2 = g_2^m + a_2 \pmod r \]
\[ b_1 = g_1^m + a_1 \pmod r \quad (2.17) \]

Since \( 0 \leq \nu < r^{\nu-1} \), we shall search \( \nu \) in the form \( \nu = (l_{\nu-1} \cdots l_1)_{r} \), where \( l_1, \cdots, l_{\nu-1} \) are unknown quantities. Then \( \nu \nu = (l_{\nu-1} \cdots l_1)_{r} \). When \( \nu \nu \leq i < (l + 1) \) then \( i = (l_{\nu-1} \cdots l_1 l_0) \), for \( 0 \leq l_0 < r \).

Let us denote \( \varphi(i) = 0, c_1 c_2 \cdots c_\nu \) where

\[ c_1 = l_0 + \sum_{h=1}^{\nu-1} l_h v_{h+1,1} \pmod r \]
\[ c_2 = l_1 + \sum_{h=2}^{\nu-1} l_h v_{h+1,2} \pmod r \]

\( c_{\nu-1} = l_{\nu-2} + l_{\nu-1} v_{\nu,\nu-1} \pmod r \)
\[ c_\nu = l_{\nu-1}. \]

Then we have

\[ \delta_1, \delta_2, \cdots, \delta_{\nu-1} \text{ are the step-by-step carries and else} \]
\[ d_\nu = g_\nu^m + c_\nu \pmod r \]
\[ d_{\nu-1} = g_{\nu-1}^m + \delta_{\nu-1} + c_{\nu-1} \pmod r \]
\[ d_2 = g_2^m + \delta_2 + c_2 \pmod r \]
\[ d_1 = g_1^m + \delta_1 + c_1 \pmod r \quad (2.18) \]

For the demonstration of the equality (2.16) we make equal the numbers, constructed in (2.17) and (2.18), and we get

\[ l_{\nu-1} \equiv l_{\nu-1} \pmod r \]
\[ l_{\nu-2} + l_{\nu-1} v_{\nu,\nu-1} + \delta_{\nu-1} \equiv l_{\nu-2} + l_{\nu-1} v_{\nu,\nu-1} \pmod r \]
\[ l_1 + \sum_{h=1}^{\nu-1} l_h v_{h+1,2} + \delta_l \equiv l_1 + \sum_{h=2}^{\nu-1} l_h v_{h+1,2} \pmod r \]
\[ l_0 + \sum_{h=1}^{\nu-1} l_h v_{h+1,1} + \delta_l \equiv l_0 + \sum_{h=1}^{\nu-1} l_h v_{h+1,1} \pmod r. \]

Since \( 0 \leq l_{\nu-1}, l_{\nu-1} < r \), then equation \( l_{\nu-1} \equiv l_{\nu-1} \pmod r \) has the only solution \( l_{\nu-1} = l_{\nu-1} \). Consecutively we solve the left over equations and get uniquely integer number \( \nu = (l_{\nu-1} \cdots l_1)_{r} \), such that \( 0 \leq \nu < r^{\nu-1} \).

Since \( l_0 \) takes \( r \) different values in the set \( \{0, 1, \ldots, r-1\} \), then and \( l_0 \) \( r \) different values in the set \( \{0, 1, \ldots, r-1\} \) and \( \nu \nu \leq i < (l + 1) \).

Finally, we establish a bijection between the sets from the two sides of the equation (2.16).

Let \( p \) and \( q \) be such that \( 0 \leq p, q < r^\nu, p \neq q \) and \( p' \) and \( q' \) are the numbers, satisfying the equality (2.16). We shall prove that \( p' \neq q' \). Let us admit that \( p' = q' = \alpha \). Then we have

\[ \{\varphi(m) : m \equiv 0 \pmod {r^\nu}, p \leq j < (p + 1) \} \]
\[ = \{\varphi(m) + \varphi(i) \pmod 1 : m \equiv 0 \pmod {r^\nu}, \alpha \leq i < (\alpha + 1) \}. \]
and
\[ \{\varphi(m + j) : m \equiv 0 \pmod{r^s}, qr \leq j < (q + 1)r\} \]
\[ = \{\varphi(m) + \varphi(i) \pmod{1}, qr \leq j < (q + 1)r\}. \]

Then we have
\[ \{\varphi(m + j) : m \equiv 0 \pmod{r^s}, pr \leq j < (p + 1)r\} \]
\[ = \{\varphi(m + j) : m \equiv 0 \pmod{r^s}, qr \leq j < (q + 1)r\}. \]

This is a contradiction, since the function \( \varphi \) is an injection; so the equation (2.16) is proved.

From (2.15) and (2.16) we get
\[ r^{v-1}\{\varphi(m + j) : m \equiv 0 \pmod{r^s}, 0 \leq j < r\} \]
\[ = r^{v-1}\{\varphi(m) + \varphi(i) \pmod{1}, qr \leq j < (q + 1)r\}. \]

The lemma is proved.

3. AN ESTIMATION FROM ABOVE FOR THE DIAPHONY OF \( P_r \)-SEQUENCES.

THEOREM 3.1. Let in the guiding matrix \( (v_{i,j}) \) every \( v_{s,j} = 1 \) and for \( j > s \) every \( v_{s,j} = 0 \) and let \( \sigma = (\varphi(n))_{n=0}^{\infty} \) be the \( P_r \)-sequence which is produced by the \( (v_{i,j}) \). Then for every positive integer \( N \) we have
\[ F_N(\sigma) \leq c(r)N^{-1}(\log((r - 1)N + 1))^{1/2}, \]
where the constant \( c(r) \) is given by
\[ c(r) = \pi((r^2 - 1)/3 \log r)^{1/2}. \]

The proof of this theorem is based on a non-trivial estimate for the trigonometric sum of an arbitrary \( P_r \)-sequence.

3.1. AN ESTIMATION OF THE TRIGONOMETRIC SUM OF ARBITRARY \( P_r \)-SEQUENCE.

Let \( X = (x_n)_{n=0}^{\infty} \) be arbitrary sequence in interval \( E.A \) trigonometric sum, \( S_N(X; h) \), of the sequence \( X \), where \( h \) is an integer is the quantity
\[ S_N(X; h) = \sum_{n=0}^{N-1} \exp(2\pi i nx_n). \]

LEMMA 3.1. Let \( N = P + Q \), where \( P \) and \( Q \) are arbitrary integers. Then for every integer \( h \) and arbitrary sequence \( X = (x_n)_{n=0}^{\infty} \) we have
\[ |S_N(X; h)| \leq |S_P(X; h)| + |S_Q(X; h)|, \]
where
\[ S_{P+Q}(X; h) = \sum_{n=0}^{P+Q-1} \exp(2\pi i x_n). \]

The proof of lemma is obvious.

LEMMA 3.2. Let \( N = a r^n \), where \( a \geq 1 \) and \( n \geq 0 \) are integers.

Then for every integer \( h \) we have
\[ |S_N(X; h)| \leq \sum_{i=1}^{a} |S_{i-1}^{r^n}(X; h)|, \]
where
\[ S_{P+Q}(X; h) = \sum_{n=0}^{P+Q-1} \exp(2\pi i x_n). \]

The proof of lemma is based of Lemma 3.1 and is done by induction on \( a \).

Let \( a \) be an arbitrary integer and \( q \) a positive integer. We define the function \( \delta_q(a) \) by
\[ \delta_q(a) = \begin{cases} \frac{1}{L} \text{ if } a \equiv 0 \pmod{q} \\ 0 \text{ if } a \not\equiv 0 \pmod{q} \end{cases} \]

It is well known that for every integer \( a \) and every natural \( q \) we have
\[ \sum_{z=1}^{q-1} \exp(2\pi i a z/q) = q \delta_q(a) \]

LEMMA 3.3. Let \( N \geq 1 \) be an integer and
\[ N = \sum_{j=0}^{\infty} a_j r^j, a_j \in \{0, 1, \ldots, r-1\} \quad (j = 0, 1, \ldots) \]
(3.2)
be its \( r \)-adic representation.

Let in the guiding matrix \( (v_{i,j}) \) every \( v_{s,s} = 1 \) and for \( j > s \) every \( v_{s,j} = 0 \) and \( \sigma = (\varphi(n))_{n=0}^{\infty} \) be the \( P_r \)-sequence which is product of \( (v_{i,j}) \).

Then for every integer \( h \) we have
\[ |S_N(\sigma; h)| \leq \sum_{j=0}^{\infty} a_j r^j \delta_{r^j}(h) \]

PROOF. Let \( N \geq 1 \) be an integer with \( r \)-adic representation of a type (3.2).
We shall prove that for every integer $h$ and for every sequence $X$ in interval $E$ we have the estimation

$$|S_N(X; h)| \leq \sum_{j=0}^{\infty} \sum_{m=1}^{a_{j}} |S_{(m-1)r^j}(X; h)|,$$

where we have the supposition that when $a_j = 0$, the inside sum is 0.

Let $h$ be an integer. For every $N \geq 1$ exists an integer $n$, such that $N < r^n$. We shall prove the lemma by the induction on $n$.

If $n = 1$, then the estimation (3.3) is trivial.

We suppose, that (3.2) is true for every integer $N$, $1 \leq N < r^n$, where $n$ is some integer.

Let now $N$ such that $r^n \leq N < r^{n+1}$. By here we have, that in (3.2) $a_j = 0$ for $j > n$. Let $N = P + Q$ where $P = a_n r^n$ and $Q = \sum_{j=0}^{n-1} a_j r^j$.

By Lemma 3.1, Lemma 3.2 and the induction supposition we get

$$|S_N(X; h)| \leq |S_{n_r^n}(X; h)| + \sum_{j=0}^{n-1} \sum_{m=1}^{a_j} |S_{(m-1)r^j}(X; h)|$$

such that (3.3) is proved. If $Q = 0$, then (3.5) is got by Lemma 3.2.

Let now $j, 0 \leq j \leq n$ be arbitrary fixed number and consider that $1 \leq m \leq a_j$. If $m = 1$, then by Lemma 2.2 for the trigonometric sum $S_{(m-1)r^j}(\sigma; h)$ we have

$$S_{(m-1)r^j}(\sigma; h) = r^j \delta_m(h).$$

(3.4)

Let now $2 \leq m \leq a_j$. Then for the trigonometric sum $S_{(m-1)r^j}(\sigma; h)$, by Lemma 2.4, we have

$$S_{(m-1)r^j}(\sigma; h) = \sum_{m=1}^{a_j} \exp(2\pi i h \varphi(n)) \sum_{k=0}^{r^{-1} \varphi((m-1)r^j)} \exp(2\pi i h \varphi(k)).$$

Thus for the module of the trigonometric sum $S_{(m-1)r^j}(\sigma; h)$ we get

$$|S_{(m-1)r^j}(\sigma; h)| = r^j \delta_m(h).$$

(3.5)

From (3.3), (3.4) and (3.5) we get

$$|S_N(\sigma; h)| \leq \sum_{j=0}^{\infty} a_j r^j \delta_{r^j}(h).$$

The lemma is proved.

3.2. PROOF OF THEOREM 3.1.

Let $(u_{x,y})$ is an arbitrary guiding matrix, such that on principal diagonal there stand ones, and over him zeros and $\sigma = (\varphi(n))_{n=0}^{\infty}$ is $P_r$-sequence, which is bred by the matrix $(u_{x,y})$.

We choose $N \geq 1$ arbitrary integer and let has $r$-adic representation in the form

$$N = \sum_{j=1}^{k} a_j r^j \in \{1, \ldots, r-1\}, j = 1, 2, \ldots, k),$$

where

$$0 \leq n_1 < n_2 < \ldots < n_k.$$

are integer numbers.

From Lemma 3.3 for every integer $h$ we have

$$|S_N(\sigma; h)| \leq \sum_{j=1}^{k} a_j r^{n_j} \delta_{r^n}(h) \leq (r-1) \sum_{j=1}^{k} r^{n_j} \delta_{r^n}(h).$$

By the last inequality for the diaphony $F_N(\sigma)$ of $\sigma$ we have

$$(N F_N(\sigma))^2 = 2(2)^{\infty} \sum_{h=1}^{\infty} h^{-2} |S_N(\sigma; h)|^2 \leq$$

$$2(r-1)^2 \sum_{h=1}^{\infty} h^{-2} \sum_{j=1}^{k} r^{n_j+n_v} \delta_{r^v}(h) \delta_{r^n}(h) =$$

$$2(r-1)^2 \sum_{j=1}^{k} \sum_{v=1}^{\infty} r^{n_j+n_v} \sum_{h=1}^{\infty} h^{-2} \delta_{r^v}(h) \delta_{r^n}(h).$$

(3.7)

If the matrix $|a_{j,v}| |(1 \leq j, v \leq k)$ is symmetric then we have

$$\sum_{j=1}^{k} \sum_{v=1}^{\infty} a_{j,v} = 2 \sum_{j=1}^{k} \sum_{v=1}^{\infty} a_{j,v} = \sum_{j=1}^{k} a_{j,j}.$$

By here and (3.7) we get

$$(N F_N(\sigma))^2 \leq 4(r-1)^2 \sum_{j=1}^{k} \sum_{v=1}^{\infty} r^{n_j+n_v} \sum_{h=1}^{\infty} h^{-2} \delta_{r^v}(h) \delta_{r^n}(h)$$

$$- 2(r-1)^2 \sum_{j=1}^{k} r^{n_j} \sum_{h=1}^{\infty} h^{-2} \delta_{r^n}(h).$$

(3.8)

For $j$ and $v$ such that $1 \leq v \leq j \leq k$, we have

$$\delta_{r^v}(h) \delta_{r^n}(h) = \delta_{r^v}(h).$$

(3.9)
for every integer $h$.

Beside this we have
\[ \sum_{k=1}^{\infty} h^{-2} \delta_{\gamma_k}^2 (h) = \pi^2 / 6 \mu^{2n_i}. \] (3.10)

By (3.8), (3.9) and (3.10) we have
\[ (NF_N (\sigma))^2 \leq \left(2 \pi^2 (r-1)^2 / 3 \right) \sum_{j=1}^{k} \sum_{i=1}^{j} r^n - n_j - (\pi^2 / 3)(r-1)^2 k \] (3.11)

For the sum in last equality holds, that
\[ \sum_{j=1}^{k} \sum_{i=1}^{j} r^n - n_j = \sum_{j=1}^{k} r^n \sum_{j=0}^{k} r^{-n_j} < \sum_{j=1}^{k} r^n \sum_{n_0}^{\infty} r^{-n} = (rk)/(r-1) \] (3.12)

From (3.11) and (3.12) we have
\[ (NF_N (\sigma))^2 \leq \left(\pi^2 / 3 (r^2 - 1) \right) \] (3.13)

From (3.6) we get that
\[ N \geq \sum_{j=1}^{k} r^n_j \geq \sum_{j=0}^{k-1} r^j = (r^k - 1)/(r-1). \]

Thus we discover
\[ k \leq \left(\log((r-1)N) + 1\right) / \log r \] (3.14)

From (3.13) and (3.14) we have
\[ F_N (\sigma) \leq \pi ((r^2 - 1) / 3 \log r)^{1/2} N^{-1} (\log((r-1)N + 1))^{1/2}. \]

Theorem 3.1 is proved.

In the case, where the guiding matrix $(v_n)$ is a unit matrix $I$, the sequence which is bred by $I$ is called Van der Corput-Halton's sequence. In 1935 it was first introduced by Van der Corput [18] and generalized in 1960 by Halton [19].

In this case the operation $^*$ turns out to be a simple addition.

By $\varphi_r (i) (i = 0, 1, \ldots)$ we signify the general term of the Van der Corput-Halton-sequence.

For $r = 2$ the sequence of general terms $\varphi_2 (i) (i = 0, 1, \ldots)$ is called Van der Corput-sequence.

By Theorem 3.1 we can get the following corollaries.

COROLLARY 3.1. Let $\sigma = (\varphi_r (i))_{i=0}^{\infty}$ be the Van der Corput-Halton-sequence. Then for every positive integer $N$, we have
\[ F_N (\sigma) \leq c(r) N^{-1} (\log((r-1)N) + 1))^{1/2}, \]
where the constant $c(r)$ is determined by the equality (3.1).

COROLLARY 3.2. Let $\sigma = (\varphi (i))_{i=0}^{\infty}$ be the Van der Corput-sequence. Then for every $N \geq 1$ we have
\[ F_N (\sigma) < 4N^{-1} (\log N)^{1/2}. \]

COROLLARY 3.3. Let $\sigma = (\varphi (i))_{i=0}^{\infty}$ be arbitrary binary $P_r$-sequence. Then
\[ \lim_{N \to \infty} (NF_N (\sigma))/ (\log N)^{1/2} \leq \pi / (\log 2)^{1/2} = 3, 7773 \ldots. \]

We note that the Corollary 3.1 and Corollary 3.2 are announced without proof by Proinov and Grozdanov [20] and proved by Proinov and Grozdanov [21].

4. ON QUADRATIC DISCREPANCY OF THE SYMMETRIC SEQUENCE PRODUCED BY THE ARBITRARY $P_r$-SEQUENCE.

In this section, we given an application of Theorem 3.1 to the problem of finding infinite sequences in $E$, with the best possible order of magnitude for the quadratic discrepancy.

We need the notion of symmetric sequence (see [11]). A sequence $\sigma = (x_n)_{n=0}^{\infty}$ in $E$ is called symmetric if for every integer $n \geq 0$ we have $x_{2n} + x_{2n+1} = 1$. A symmetric sequence $\sigma' = (b_n)_{n=0}^{\infty}$ in $E$ is said to be produced by an infinite sequence $\sigma = (a_n)_{n=0}^{\infty}$, if for every integer $n \geq 0$ we either have $a_n = b_{2n}$ or $a_n = b_{2n+1}$. Obviously, every infinite sequence in $E$ produce at least one symmetric sequence.

By Sobol ([5], p. 117) it is clear that the exact order of quadratic discrepancy of $P_2$-sequence is $O(N^{-1} \log N)$.

We shall prove that the quadratic discrepancy of arbitrary symmetric sequence, which is produced by arbitrary $P_r$-sequence has exact order $O(N^{-1} (\log N)^{1/2})$. In the foundation of this problem stands Theorem A, proved by Proinov and Grozdanov [20].
By this and Theorem 3 1 follows

**THEOREM 4**  Let \( \varphi \) be an arbitrary symmetric sequence in \( E \), which is produced by an arbitrary \( P_r \)-sequence. Then for every integer \( N \geq 2 \) we have

\[
T_N(\varphi) < c(r)N^{-1} (\log(r - 1)N)^{1/2} + N^{-1},
\]

where \( c(r) \) is defined by the equality (3 1).

From Theorem 4 1 for the case \( r = 2 \) we have

\[
\lim_{N \to \infty} NT_N(\varphi)/(\log(N))^{1/2} \leq 1/(\log 2)^{1/2} = 1, 201 \ldots,
\]

for every symmetric sequence \( \varphi \) produced by the \( P_2 \)-sequence.

We note that Faure [22] proved that for the symmetric sequence \( \varphi \), produced by the Van der Corput-sequence, the constant \( \lim_{N \to \infty} (NT_N(\varphi)/(\log(N))^{1/2} \) is between 0, 298 and 0, 321.

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