ELLIPITC RIESZ OPERATORS ON THE WEIGHTED SPECIAL ATOM SPACES

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ABSTRACT. In this paper we study the boundedness and convergence of \( \sigma^s_r(f) \) and \( \tilde{\sigma}^s_r(f) \), the elliptic Riesz operators and the conjugate elliptic Riesz operators of order \( s > 0 \), on the weighted special atom space \( B(\omega) \).

KEY WORDS AND PHRASES. Elliptic Riesz operators, weighted special atom space, Lorentz spaces.

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1. INTRODUCTION.

Let \( R^n \) be \( n \)-dimensional Euclidean space and \( Z^n \) be the unit lattice in \( R^n \). The \( n \)-Torus \( T^n \) is the coset space \( R^n/(2\pi Z^n) \), \( Q^n = \{ x = (x_1, \ldots, x_n) : 0 < x_k \leq 2\pi, 1 \leq k \leq n \} \). Let \( A(D) \) be a self-adjoint elliptic differential operator with real coefficients defined on \( C^\infty_0(R^n) \), \( A(D) = \sum a_\alpha D^\alpha \), where \( D^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is multi index and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). We always assume that the set \( \{ x \in R^n : A(x) < 1 \} \) is convex and its boundary has non-vanishing Gaussian curvature everywhere.

The elliptic Riesz operators and the conjugate elliptic Riesz operators of order \( s > 0 \) are defined respectively by

\[
\sigma^s_r(f, x) = \sum_{m \in Z^n} (1 - A(m/r))^s \hat{f}(m) e^{imx}
\]

(1.1)

\[
\tilde{\sigma}^s_r(f, x) = \sum_{m \in Z^n} (1 - A(m/r))^s \hat{f}(m) \tilde{K}(m) e^{imx}
\]

(1.2)

where

\[
\hat{f}(m) = (2\pi)^{-n} \int_{Q^n} f(x) e^{-imx} \, dx
\]

are the multiple Fourier coefficients of \( f \), \( K(x) = P(x)/|x|^{n+m}(x \neq 0) \) is a kernel with a homogeneous and harmonic polynomial \( P(x) \) of order \( m \), and \( \tilde{f} \) is the conjugate function of \( f \) with respect to the kernel \( K(x) \). \( \beta_+ = \max\{0, \beta\} \). If \( A(\xi) = |\xi|^2 \), \( \sigma^s_r(f) \), \( \tilde{\sigma}^s_r(f) \) is just the usual Bochner-Riesz mean.

The maximal elliptic Riesz operators defined by

\[
\sigma^s(f, x) = \sup_{r > 0} |\sigma^s_r(f, x)|, \tilde{\sigma}^s(f, x) = \sup_{r > 0} |\tilde{\sigma}^s_r(f, x)|.
\]

In this paper, using the weighted special atom space \( B(\omega) \), we will study the boundedness and convergence of \( \sigma^s(f) \) and \( \tilde{\sigma}^s(f) \) for all \( s > 0 \) and \( n = 1 \).
We rewrite $B(\omega)$ which was introduced in [4]:

$$B(\omega) = \left\{ f : T \to \mathbb{R}, f(t) = \sum_{k=0}^{\infty} C_k b_k(t), \sum_{k=0}^{\infty} |C_k| < \infty \right\},$$

each $b_k$ is a weighted special atom, that is, a real valued function $b$, defined on $T = [0, 2\pi]$, which is either $b(t) = 1/(2\pi)$ or $b(t) = \omega(|Q|)^{-1/q}$. $[\chi_R(t) - \chi_L(t)]$, $1 \leq q < \infty$, where $Q$ is an interval in $T$, $L$ is the left half of $Q$ and $R$ is the right half, $|Q|$ denotes the length of $Q$, $\chi_Q$ the characteristic function of $Q$ and $\omega$ is a non-negative real valued function which is increasing, and $\omega(0) = 0$. $B(\omega)$ is endowed with the norm $\|f\|_{B(\omega)} = \inf\left\{ \sum_{k=0}^{\infty} |C_k| \right\}$, where the infimum is taken over all possible representations of $f$. $B(\omega)$ is a Banach space.

A function $\omega : [0, \infty) \to [0, \infty)$ is said to be in the class $b_\lambda (0 < \lambda < \infty)$, if it satisfies

1. $\omega(0) = 0$,
2. $\omega$ is non-decreasing,
3. $\omega(t)/t$ is decreasing,
4. $\int_0^\infty \omega(t)/tdt \leq C\omega(h)$, $C$ an absolute constant,
5. $\int_0^{2\pi} \omega(t)/t^{1+\beta}dt \leq C\omega(h)/h^\beta$ with $C$ independent of $h$ and $\omega$.

Example of functions in the class $b_\lambda$ are $\omega(t) = t^\alpha (0 < \alpha < 1)$ and $\omega(t) = t^\alpha(\log(e/t))^\beta$, $0 < \alpha, \beta \geq 0$.

We also define the space $L(\phi)$ be $L(\phi) = \{ f : T \to \mathbb{R}, \|f\|_\phi < \infty \}$, where $\|f\|_\phi = (\int_T |f^*(t)|^q\phi(t)dt)^{1/q}$, $1 \leq q < \infty$ and $f^*$ is the decreasing rearrangement of $f$, defined by $f^*(t) = \inf\{y : |\{x : |f(x)| > y\}| \leq t\}$, the outside bars means the Lebesgue measure of the set $\{x : |f(x)| > y\}$, $\phi$ is a non-negative decreasing function. $\|f\|_\phi$ is a norm if and only if $\phi$ is a non-negative decreasing function. $L(\phi)$ is a Banach space. If $\omega(t) = (q/p)t^{q/p}$, $1 \leq q \leq p < \infty$, $\phi(t) = \omega(t)/t$, then the space $L(\phi)$ is the Lorentz space $L(p, q)$ in [6,7].

The main result of this paper is stated as follows:

**THEOREM 1.** Suppose $\omega \in b_\lambda$, $1 \leq \lambda < \infty$, $\phi(t) = \omega(t)/t$, then $\sigma^s(f)$ is of type $(B(\omega), L(\phi))$ for all $s > 0$, that is,

$$\|\sigma^s(f)\|_\phi \leq C\|f\|_{B(\omega)}, \quad f \in B(\omega).$$

**COROLLARY 1.** Suppose $\omega \in b_\lambda$, $1 \leq \lambda < \infty$, and $f \in B(\omega)$, then $\sigma^s(f, x)$ converges to $f(x)$ almost everywhere for all $s > 0$.

**THEOREM 2.** Suppose $\omega \in b_\lambda$, $1 \leq \lambda < \infty$, $\phi(t) = \omega(t)/t$, then $\sigma^s(f)$ is of type $(B(\omega), L(\phi))$ for all $s > 0$, that is,

$$\|\sigma^s(f)\|_\phi \leq C\|f\|_{B(\omega)} \leq C\|f\|_{B(\omega)}, \quad f \in B(\omega).$$

**COROLLARY 2.** Suppose $\omega \in b_\lambda$, $1 \leq \lambda < \infty$, and $f \in B(\omega)$, then $\sigma^s(f, x)$ converges to $f(x)$ almost everywhere for all $s > 0$.

**REMARK 1.** When $n = 1$, $A(\xi) = |\xi|^2$, $\sigma^s_\lambda(f, x)$ become

$$\sigma^s_\lambda(f, x) = \sum_{|k| < r} (1 - (|k|/r)^2)^s f^*(k)e^{ikx}. \quad (1.3)$$

As $s \to 0$, (1.3) become the partial sums of Fourier series of $f$, when $s = 1/2$, (1.3) are essentially equivalent to the classical Cesàro means. Consequently, the main result in [5,6] become a special case of our results.

**REMARK 2.** For the maximal $(C, \alpha)$ operators $T$ are defined by

$$T(f, x) = \sup_{r} \|\sigma^n_\alpha(f, x)\| \quad (1.4)$$

where
Since \((C, \alpha)\) kernels
\[
K_\alpha^n(t) = \sum_{k=0}^{n} A_{n-k} D_k(t) / A_n
\]
satisfies
\[
|K_\alpha^n(t)| \leq \begin{cases} 
\frac{A_n}{(1 + nt)(1 + (nt)^\alpha)} & 0 < \alpha < 1, 0 \leq t \leq \pi, \\
\frac{C}{|t|} & \alpha = 1, 0 < |t| \leq \pi,
\end{cases}
\]
thus using the same methods for \(\phi(t) = \omega(t)/t\) we can prove
\[
\|Tf\|_\phi \leq C\|f\|_{B(\omega)}, \quad f \in B(\omega), \quad 0 < \alpha < 1.
\]

2. PROOFS OF THEOREMS

PROOF OF THEOREM 1. Let \(f^\alpha(x) = f(x - \alpha)\), then the operator \(T_\alpha f = f^\alpha\) is of type \((B(\omega), B(\omega))\). Consequently, we just need to prove the result for \(f_\alpha(t) = [\omega(2h)]^{-1/q} [\chi_{[-h,0]}(t) - \chi_{[0,h]}(t)], h > 0\) which will follow from the estimate for \(g(t) = \chi_{[0,h]}(t)\). Let \(H(x) = (2\pi)^{-1} \int_{R} (1 - A(y))_+ e^{iyx} dy, s > 0, H_{1/r}(x) = rH(rx)\), then \(\sigma_r^\alpha(f, x) = (f*K_{1/r})(x)\), where
\[
K_{1/r}(x) = \sum_{k=-\infty}^{\infty} H_{1/r}(x + 2k\pi).
\]

We may assume \(r > 1\). By the inequality (see [2]):
\[
|H(x)| \leq C(1 + |x|)^{-s-1},
\]
we get
\[
|K_{1/r}(x)| \leq Cr \sum_{k=-\infty}^{\infty} (1 + r|2k\pi + x|)^{-(s+1)} \leq Cr(1 + r|x|)^{-(s+1)}.
\]

Thus
\[
|\sigma_r^\alpha(g, x)| = \left| \int_{-\infty}^{x} g(y) K_{1/r}(x - y) dy \right| = \left| \int_{0}^{h} K_{1/r}(x - y) dy \right| \leq \int_{-h}^{x} |K_{1/r}(t)| dt
\]
\[
\leq C \int_{-h}^{x} r(1 + rt)^{-1} dt \leq Ch(x - h)^{-1} < 2Ch/x,
\]
for \(x > 2h\), and \(|\sigma_r^\alpha(g, x)| \leq -2Ch/x\) for \(x < -2h\). On the other hand, we have
\[
|\sigma_r^\alpha(g, x)| \leq \int_{0}^{h} |K_{1/r}(x - y)| dy \leq \int_{-h}^{x} \left( \sum_{k=-\infty}^{\infty} |H_{1/r}(x + 2k\pi - y)| \right) dy
\]
\[
= \int_{-\infty}^{\infty} |H_{1/r}(y)| dy \leq C \int_{-\infty}^{\infty} (1 + |t|)^{-(s+1)} dt < \infty.
\]

Consequently, we have
\[
|\sigma^\alpha(g, x)| \leq \begin{cases} 
A, & \text{for all } x, \\
2Ch/|x|, & \text{for } |x| > 2h.
\end{cases}
\]

Let \(\phi(t) = \omega(t)/t\). By (2.1) and the conditions on \(\omega\), we get
\[
\|\sigma^s(g)\|_\phi^q = \int_0^{2\pi} ((\sigma^s(g))^s(x))^{q} \omega(x)/zdx \leq A^q \int_0^{2h} \omega(x)/zdx \\
+ (2Ch)^q \int_0^{2h} \omega(x)/x^{(q+1)} dx \leq C A^q \omega(2h) + (2Ch)^q (\omega(2h)/(2h)^q) = C \omega(2h).
\]

The constant \( C \) may not be the same at every occurrence in this paper. Thus \( \|\sigma^s(f_h)\|_\phi \leq 2\omega(2h)^{-1/q^s}\|\sigma^s(g)\|_\phi \leq C \) and so if \( f \in B(\omega) \), then \( f(t) = \sum_{k=0}^\infty C_k b_k(t) \), where

\[
b_k(t) = \omega(Q_k)^{-1/q} [\chi_{R_k}(t) - \chi_{\ell_k}(t)]
\]

and \( \sum_{k=0}^\infty |C_k| < \infty \), we have \( \|\sigma^s(f)\|_\phi \leq C \sum_{k=0}^\infty |C_k| \), which implies \( \|\sigma^s(f)\|_\phi \leq C \|f\|_{B(\omega)} \). The proof is complete.

**PROOF OF COROLLARY 1.** Let

\[
\omega(f, x) = \limsup_{r \to \infty} \left| \sigma_{r_1}^s(f, x) - \sigma_{r_2}^s(f, x) \right|, r_1, r_2 > r, \tag{2.2}
\]

then \( \omega(f, x) \leq 2\sigma^s(f, x) \) and so

\[
\|\omega(f)\|_\phi^q = \int_0^{2\pi} ((\omega(f))^s(x))^{q} \omega(x)/zdx \leq 2 \int_0^{2\pi} ((\sigma^s(f))^s(x))^{q} \omega(x)/zdx \\
= 2\|\sigma^s(f)\|_\phi^q. \tag{2.3}
\]

Since \( f \in B(\omega) \), then \( f(x) = \sum_{k=0}^\infty C_k b_k(x) \), where \( \sum_{k=0}^\infty |C_k| < \infty \) and the \( b_k \) are weighted special atoms.

By Theorem 1 and (2.3), \( \sigma^s(f) \in L(\phi) \) which implies \( \omega(f) \in L(\phi) \) for \( \phi(t) = \omega(t)/t \). On the other hand, we see that \( \omega(f) = \omega(f - f_m) \) where \( f_m(x) = \sum_{k=0}^m C_k b_k(x) \) and \( \|f_m - f\|_{B(\omega)} \to 0 \) as \( m \to \infty \). Then

\[
\omega(f, x) = \omega(f - f_m, x) \leq 2\sigma^s(f - f_m, x).
\]

By Theorem 1,

\[
\|\omega(f)\|_\phi \leq 2\|\sigma^s(f - f_m)\|_\phi \leq 2C\|f - f_m\|_{B(\omega)}.
\]

So letting \( m \to \infty \), we get \( \|\omega(f)\|_\phi = 0. \) Thus \( \omega(f, x) = 0 \) almost everywhere, which implies \( \sigma_{r_1}^s(f, x) \) converges to \( f(x) \) almost everywhere. The proof is complete.

Let \( f \in B(\omega) \), then \( f(x) = \sum_{k=0}^\infty C_k b_k(x) \), where \( \sum_{k=0}^\infty |C_k| < \infty \) and the \( b_k \) are weighted special atoms. Thus \( \tilde{f}(x) = \sum_{k=0}^\infty C_k b_k(x) \) and so \( \|\tilde{f}\|_{B(\omega)} \leq \|f\|_{B(\omega)}. \) Now using \( \sigma_{r_1}^s(f, x) = \sigma_{r_1}^s(\tilde{f}, x) \) and Theorem 1, we can similarly show that Theorem 2 and Corollary 2. The details will be omitted.

**REMARK 3.** Theorem 1 and 2 are also true if we replace the above \( \omega(|Q|) \) by a weight \( \omega(Q) = \int_0^\omega \omega(x)/dx \), where \( \omega \in A_\infty \), the proofs are the same.

**REFERENCES**


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