THE REGULAR OPEN-OPEN TOPOLOGY FOR FUNCTION SPACES

KATHRYN F. PORTER

Department of Mathematical Sciences
Saint Mary's College of California
Moraga, CA 94575

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ABSTRACT. The regular open-open topology, T_{roo} , is introduced, its properties for spaces of continuous functions are discussed, and T_{roo} is compared to T_{oo} , the open-open topology. It is then shown that T_{roo} on $H(X)$, the collection of all self-homeomorphisms on a topological space, (X, T) , is equivalent to the topology induced on $H(X)$ by a specific quasi-uniformity on X , when X is a semi-regular space.

KEY WORDS AND PHRASES. Compact-open topology, admissible topology, open-open topology, quasi-uniformity, regular open set, semi-regular space, topological group.

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1. INTRODUCTION.

A set-set topology is one which is defined as follows: Let (X, T) and (Y, T^*) be topological spaces. Let $\mathbf{U}$ and $\mathbf{V}$ be collections of subsets of X and Y , respectively. Let $F \subset Y^X$, the collection of all functions from X into Y . Define, for $U \in \mathbf{U}$ and $V \in \mathbf{V}$, $(U, V) = \{f \in F : f(U) \subset V\}$. Let $S(\mathbf{U}, \mathbf{V}) = \{(U, V) : U \in \mathbf{U} \text{ and } V \in \mathbf{V}\}$. If $S(\mathbf{U}, \mathbf{V})$ is a subbasis for a topology $T(\mathbf{U}, \mathbf{V})$ on F then $T(\mathbf{U}, \mathbf{V})$ is called a set-set topology.

Some of the most commonly discussed set-set topologies are the compact-open topology, T_{co} , which was introduced in 1945 by R. Fox [1], and the point-open topology, T_p . For T_{co} , $\mathbf{U}$ is the collection of all compact subsets of X and $\mathbf{V} = T^*$, the collection of all open subsets of Y , while for T_p , $\mathbf{U}$ is the collection of all singletons in X and $\mathbf{V} = T^*$.

In section 2 of this paper, we shall introduce and discuss the regular open-open topology for function spaces. It will be shown which of the desirable properties T_{roo} possesses. In section 3, Pervin and almost-Pervin spaces are explained.

The fact that T_{roo} , on $H(X)$, is actually equivalent to the regular-Pervin topology of quasi-uniform convergence will be discussed in section 4 along with the topic of quasi-uniform convergence. The advantage of the regular open-open topology is the set-set notation which provides us with

simple notation and, hence, our proofs are more concise than those using the cumbersome notation of the quasi-uniformity.

We assume a basic knowledge of quasi-uniform spaces. An introduction to quasi-uniform spaces may be found in Fletcher and Lindgren's [2] or in Murdeshwar and Naimpally's book [3].

Throughout this paper we shall assume (X, T) and (Y, T^*) are topological spaces.

2. THE REGULAR OPEN-OPEN TOPOLOGY.

A subset, W , of X is called a regular open set provided $W = \text{Int}_X(Cl_X(W))$. If we let $\mathbf{U}$ be the collection of all regular open sets in X and $\mathbf{V} = T^*$, then $S_{r_{oo}} = S(\mathbf{U}, \mathbf{V})$ is the subbasis for a topology, $T_{r_{oo}}$, on any $F \subset Y^X$, which is called the regular open-open topology.

A topological space, X , is called semi-regular provided that for each $U \in X$ and each $x \in U$ there exists a regular open set, V , in X , such that $x \in V \subset U$. One can easily show that if (X, T) is a semi-regular space then $T_{r_{oo}} \subset T_{oo}$, the open-open topology (Porter, [4]) which has as a subbasis the set $S_{oo} = \{(U, V) : U \in T \text{ and } V \in T^*\}$.

We now examine some of the properties of function spaces the regular open-open topology possesses. The first two theorems also hold for the open-open topology even when X is not semi-regular. The proofs of these two theorems are straightforward and are left to the reader.

THEOREM 1. Let (X, T) be a semi-regular space and $F \subset C(X, Y)$. If (Y, T^*) is T_i for $i = 0, 1, 2$, then $(F, T_{r_{oo}})$ is T_i for $i = 0, 1, 2$.

A topology, T' , on $F \subset Y^X$ is called an admissible (Arens [5]) topology for F provided the evaluation map, $E: (F, T') \times (X, T) \rightarrow (Y, T^*)$, defined by $E(f, x) = f(x)$, is continuous.

THEOREM 2. If $F \subset C(X, Y)$ and X is semi-regular, then $T_{r_{oo}}$ is admissible for F .

Arens also has shown that if T' is admissible for $F \subset C(X, Y)$, then T' is finer than T_{co} . From this fact and Theorem 2, it follows, as it does for T_{oo} , that $T_{co} \subset T_{r_{oo}}$ when X is semi-regular.

THEOREM 3. The sets of the form (U, V) where both U and V are regular open sets in X form a subbasis for $(H(X), T_{r_{oo}})$.

PROOF. Let (U, V) be a subbasic open set in $(H(X), T_{r_{oo}})$. i.e., U is a regular open set and O is an open set, not necessarily regular. Let $f \in (U, O)$. Then $f(U) \subset O$, so $f \in (U, f(U)) \subset (U, O)$ and $f(U)$ is a regular open set.

Let $(G, \circ)$ be a group such that (G, T) is a topological space, then (G, T) is a topological group provided the following two maps are continuous. (1) $m: G \times G \rightarrow G$ defined by $m(g_1, g_2) = g_1 \circ g_2$ and $\Phi: G \rightarrow G$ defined by $\Phi(g) = g^{-1}$. If only the first map is continuous, then we call (G, T) a quasi-topological group (Murdeshwar and Naimpally [3]).

Note that $H(X)$ with the binary operation $\circ$, composition of functions, and identity element e , is a group. It is not difficult to show that if (X, T) is a topological space and G is a subgroup of $H(X)$ then (G, T_{oo}) is a quasi-topological group. However, (G, T_{oo}) is not always a topological group (Porter, [4]) since Φ is not always continuous although m is always continuous. But we discover the following about the regular open-open topology.

THEOREM 4. Let X be a semi-regular space and let G be a subgroup of $H(X)$. Then $(G, T_{r_{oo}})$ is a topological group.

PROOF. Let X be a semi-regular space and let G be a subgroup of $H(X)$. Let (U, V) be a subbasic open set in $T_{r_{oo}}$ such that both U and V are regular open sets. Let $(f, g) \in m^{-1}((U, V))$. Then, $f \circ g(U) \subset V$ and $g(U) \subset f^{-1}(V)$. So, $(f, g) \in (g(U), V) \times (U, g(U)) \in T_{r_{oo}} \times T_{r_{oo}}$. But $(g(U), V) \times (U, g(U)) \subset m^{-1}((U, V))$. Thus, m is continuous.

Note that the inverse map $\Phi : G \rightarrow G$ is bijective and that $\Phi^{-1} = \Phi$. Thus, in order to show that Φ is continuous, it suffices to show that Φ is an open map. To this end, let (O, U) be a subbasic open set in $T_{r_{oo}}$ where O and U are both regular open sets. Clearly, $\Phi((O, U)) = ((X \setminus U), (X \setminus O))$ since we are dealing with homeomorphisms. Note that if C, K are regular closed sets then $\text{Int}_X C, \text{Int}_X K$ are regular open sets. Thus, since $(X \setminus O), (X \setminus U)$ are regular closed sets, $\text{Int}_X(X \setminus U), \text{Int}_X(X \setminus O)$ are regular open sets. Again, since G is a set of homeomorphisms, $(X \setminus U, X \setminus O) = (\text{Int}_X(X \setminus U), \text{Int}_X(X \setminus O))$ but this is in $T_{r_{oo}}$. Therefore, $\Phi(O, U)$ is an open set in $T_{r_{oo}}$. So, Φ is open and we are done.

3. PERVIN AND ALMOST-PERVIN SPACES.

A topological space, (X, T) , is called a Pervin space (Fletcher [4]) provided that for each finite collection, $\mathcal{A}$, of open sets in X , there exists some $h \in H(X)$ such that $h \neq e$ and $h(U) \subset U$ for all $U \in \mathcal{A}$. A topological space, (X, T) is called almost-Pervin provided that for each finite collection, $\mathcal{A}$, of regular open sets, there exists some $h \in H(X)$ such that $h \neq e$ and $h(O) \subset O$ for all $O \in \mathcal{A}$.

Topologies are rarely interesting if they are the trivial or discrete topology. We have previously shown (Porter, [4]) that $(H(X), T_{oo})$ is not discrete if and only if (X, T) is a Pervin space. The situation for $T_{r_{oo}}$ is similar.

THEOREM 5. $(H(X), T_{r_{oo}})$ is not discrete if and only if (X, T) is almost-Pervin.

PROOF. First, assume that (X, T) is an almost-Pervin space. Let W be a basic open set in $T_{r_{oo}}$ which contains e ; i.e. $W = \bigcap_{i=1}^n (O_i, U_i)$ where $O_i \subset U_i$ for each $i = 1, 2, 3, \dots, n$ and O_i and U_i are regular open sets in X . $\{O_i : i = 1, 2, 3, \dots, n\}$ is a finite collection of regular open sets in X , and X is an almost-Pervin space, hence, there exists some $h \in H(X)$ such that $h \neq e$ and $h(O_i) \subset O_i \subset U_i$. So, $h \in W$ and $h \neq e$. Therefore, $(H(X), T_{r_{oo}})$ is not a discrete space.

Now assume that $(H(X), T_{r_{oo}})$ is not discrete. Let V be a finite collection of regular open sets in X . Let $O = \bigcap_{U \in V} (U, U)$. Then, O is a basic open set in $(H(X), T_{r_{oo}})$ which is not a discrete space. Hence, there exists $h \in O$ with $h \neq e$. So, (X, T) is almost-Pervin.

The above proof, along with the few needed definitions involving $T_{r_{oo}}$, is an example of the simplification that the definition of $T_{r_{oo}}$ offers over the quasi-uniform definition and notation.

4. QUASI-UNIFORM CONVERGENCE.

Recall that if Q is a quasi-uniformity on X , then the topology, T_Q , on X , which has as its

neighborhood base at x , $B_x = \{U[x] : U \in Q\}$, is called the topology induced by Q . The ordered triple (X, Q, T_Q) is called a quasi-uniform space. A topological space, (X, T) is quasi-uniformizable provided there exists a quasi-uniformity, Q , such that $T_Q = T$. In 1962, Pervin [7] proved that every topological space is quasi-uniformizable by giving the following construction.

Let (X, T) be a topological space. For each $O \in T$, define the set $S_O = (O \times O) \cup ((X \setminus O) \times X)$. Let $S = \{S_O : O \in T\}$. Then S is a subbasis for a quasi-uniformity, P , for X , called the Pervin quasi-uniformity and, as is easily shown, $T_P = T$.

If we use the same basic structure as above but change the subbasis to $S = \{S_O : O \text{ is a regular open set}\}$ then the quasi-uniformity induced will be called the regular-Pervin quasi-uniformity, RP .

If (X, Q) is a quasi-uniform space then Q induces a topology on $H(X)$ called the topology of quasi-uniform convergence w.r.t. Q , as follows: For each set $U \in Q$, let us define $W(U) = \{(f, g) \in H(X) \times H(X) : (f(x), g(x)) \in U \text{ for all } x \in X\}$. Then, $B(Q) = \{W(U) : U \in Q\}$ is a basis for Q^* , the quasi-uniformity of quasi-uniform convergence w.r.t. Q (Naimpally [8]). Let T_{Q^*} denote the topology on $H(X)$ induced by Q^* . T_{Q^*} is called the topology of quasi-uniform convergence w.r.t. Q^* . If P is the Pervin quasi-uniformity on X , T_{P^*} is the Pervin topology of quasi-uniform convergence and if RP is the regular-Pervin quasi-uniformity on X , then T_{RP^*} is called the regular-Pervin topology of quasi-uniform convergence, T_{RP^*} .

It has been shown that the open-open topology is equivalent to the Pervin topology of quasi-uniform convergence (Porter, [4]). It is also true that the regular open-open topology is equivalent to the regular-Pervin topology of quasi-uniform convergence. The method of two proofs are exactly the same and we leave this one for the reader.

THEOREM 6. Let (X, T) be a topological space and let G be a subgroup of $H(X)$. Then, $T_{\tau oo} = T_{RP^*}$ on G .

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