A CHARACTERIZATION OF POINT SEMIUNIFORMITIES

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ABSTRACT. The concept of a uniformity was developed by A. Weil and there have been several generalizations. This paper defines a point semiuniformity and gives necessary and sufficient conditions for a topological space to be point semiuniformizable. In addition, just as uniformities are associated with topological groups, a point semiuniformity is naturally associated with a semicontinuous group. This paper shows that a point semiuniformity associated with a semicontinuous group is a uniformity if and only if the group is a topological group.

KEY WORDS AND PHRASES: Uniformity, point semiuniformity, vicinities, point semi-uniformizable, homogeneous, topological group, semicontinuous group, semifundamental system, point regular, bihomogeneous.

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1. Introduction. In 1937, A. Weil [1] generalized the concept of a metric space by defining a topology-generating structure called a uniformity. There have been several generalizations of uniformities. For example, a semiuniformity, $\mathcal{U}$, for a set X is a filter of supersets of the diagonal in $X \times X$ such that for each U in $\mathcal{U}$, there is a V in $\mathcal{U}$ such that $V^{-1} = \{(y, x) | (x, y) \in V\} \subseteq U$. As with a uniformity and its other generalizations, there is a natural way to try to construct neighborhoods of points. Namely, for each x in X and U in $\mathcal{U}$ we define a slice, $U[x]$, to be $\{y | (x, y) \in U\}$. For a semiuniformity, the collection $\{U[x]\}$ does generate a topology on X but we are left with the unsatisfactory situation that some of the slices are not neighborhoods in this topology. In [2], W. Page gets around this problem by calling a semiuniformity a t -semiuniformity (for topological semiuniformity) if all the slices turn out to be neighborhoods, and he proves that a space is t -semiuniformizable (there is a t -semiuniformity which generates the topology) if and only if the space satisfies a certain separation property. We take a different approach. We define a point semiuniformity, $\mathcal{P}$, to be a semiuniformity with the added condition that for every $S \in \mathcal{P}$ there is a $T \in \mathcal{P}$ having for each $(x, y) \in T$ a $V \in \mathcal{P}$ such that $(x, y) \circ V$ and $V \circ (x, y)$ are contained in S . We will show that the slices gotten from $\mathcal{P}$ will always be neighborhoods in the topology generated by $\mathcal{P}$ and that a space is point semiuniformizable if and only if it satisfies the same separation property referred to above.

The natural association of uniformities with topological groups, or more exactly, with a fundamental system of a topological group is well known. We show that a point semiuniformity is just as naturally associated with a semicontinuous group [3] (called semitopological groups by Bourbaki [4] and L. Fuchs [5]). A semicontinuous group is a group with a topology making inversion and left and right multiplication by single elements continuous. We show that the point semiuniformity associated with a semicontinuous group is a uniformity if and only if the group is a topological group.

2. Point Semiuniformity

We begin by formalizing our definitions. A *point semiuniformity* for a set X is a filter $\mathcal{P}$ of subsets of $X \times X$ such that $\forall S \in \mathcal{P}$,

- 1) $\Delta \subseteq S$
- 2) $\exists T \in \mathcal{P}$ such that $T^{-1} \subseteq S$
- 3) $\exists T \in \mathcal{P}$ such that for each $(x, y) \in T$ there is a $V \in \mathcal{P}$ with $V \circ (x, y) \subseteq S$ and $(x, y) \circ V \subseteq S$.

A pair $(X, \mathcal{P})$ consisting of a set X and a point semiuniformity $\mathcal{P}$ on $X \times X$ is called a *point semiuniform space*. We call β a *base* for a point semiuniformity $\mathcal{P}$ if and only if the collection of all supersets of elements of β is $\mathcal{P}$. It is clear that any filter base satisfying the three conditions above is a base for a point semiuniformity.

THEOREM 1. Let $\mathcal{P}$ be a point semiuniformity for a set X and let $\mathcal{P}_x = \{S[x] \mid S \in \mathcal{P}\}$. Then $\{\mathcal{P}_x \mid x \in X\}$ forms a neighborhood base for a topology τ on X .

PROOF. Clearly, for all $S, T \in \mathcal{P}$, $x \in S[x]$ and $S[x] \cap T[x] = (S \cap T)[x]$. Now let $S[x] \in \mathcal{P}_x$. Since $S \in \mathcal{P}$ then $\exists T \in \mathcal{P}$ with the property that for each $(a, b) \in T$, $\exists U_{(a, b)} \in \mathcal{P}$ with $U_{(a, b)} \circ (a, b) \subseteq S$. In addition, since $(x, x) \in T$ then $\exists U_{(x, x)} \in \mathcal{P}$ with $U_{(x, x)} \circ (x, x) \subseteq S$. Since $U_{(x, x)} \in \mathcal{P}$ then $\exists V \in \mathcal{P}$ with the property given any $(s, t) \in V$ $\exists W_{(s, t)} \in \mathcal{P}$ with $W_{(s, t)} \circ (s, t) \subseteq U_{(x, x)}$. Now $V[x] \in \mathcal{P}_x$ and thus we must show that a neighborhood of each point of $V[x]$ is contained in $S[x]$. Suppose that $y \in V[x]$ then $(x, y) \in V$. Now $\exists W_{(x, y)} \in \mathcal{P}$ such that $W_{(x, y)} \circ (x, y) \subseteq U_{(x, x)}$. It suffices to show $W_{(x, y)}[y] \subseteq S[x]$. Therefore, let $z \in W_{(x, y)}[y]$ and so $(y, z) \in W_{(x, y)}$ which implies that $(x, z) \in W_{(x, y)} \circ (x, y) \subseteq U_{(x, x)}$. Hence, $(x, z) = (x, z) \circ (x, x) \in U_{(x, x)} \circ (x, x) \subseteq S$. Consequently, $z \in S[x]$. ■

A uniformity $\mathcal{U}$ is a semiuniformity with the property that for each $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. Thus, we see that every uniformity is a point semiuniformity and every t -semiuniformity is a point semiuniformity. We now turn our attention to which topologies can be generated by these point semiuniformities. Any topology thus induced is called a *point semiuniform topology* and the space is called a *point semiuniformizable topological space*. The finite complement topology on an infinite space is point semiuniformizable, but since the space is not completely regular, it is not uniformizable.

In [2], Page shows that a space is t -semiuniformizable if and only if $\forall x, y \in X, x \in Cl_\tau\{y\}$ iff $y \in Cl_\tau\{x\}$. We restate this closure condition in an equivalent form.

DEFINITION. A topological space (X, τ) is point regular (or p -regular) if and only if for every $V \in \tau$ and for every $x \in V, Cl_\tau\{x\} \subseteq V$.

The next proposition states some of the basic topological properties possessed by p -regular spaces.

PROPOSITION 2.

- i) Every regular space or T_1 -space is a p -regular space.
- ii) A p -regular, T_0 -space is a T_1 -space.
- iii) The continuous closed image of a p -regular space is a p -regular space, but the quotient of a p -regular space need not be p -regular.
- iv) Products and subspaces of p -regular spaces are p -regular.
- v) Although homogeneous spaces need not be p -regular, bihomogeneous spaces are p -regular.

In [2], Page shows that a space X is t -semiuniformizable if and only if it is p -regular. In his proof, he constructs a t -semiuniformity as follows: For each $x \in X$, let u_x be a neighborhood of x and let $R = U(x \times u_x)$ and $S = R \cup R^{-1}$. The collection β of all such $S \subseteq X \times X$ forms a base for a t -semiuniformity which induces the original topology τ . However, a t -semiuniformity need not be a point semiuniformity as the following example shows.

EXAMPLE 3. Consider $\mathbb{R}$, the real numbers, with the usual topology. For each $r \in \mathbb{R}$, choose neighborhoods, u_r , as follows: Let $u_1 = \mathbb{R}$. For each element of the sequence $\langle 1 - 1/n \rangle_{n=1}^\infty$, choose open interval neighborhoods so that

$$1/2 \notin u_0, \quad 0, 2/3 \notin u_{1/2}, \quad 1/2, 3/4 \notin u_{2/3}, \dots \\ (k-2)/(k-1), \quad k/(k+1) \notin u_{k-1/k}, \quad \text{etc.}$$

For $y \in \mathbb{R} - \{1, 0, 1/2, 2/3, 3/4, \dots\}$, choose any neighborhood u_y of y . Now, let $R = \bigcup_{r \in \mathbb{R}} (r \times u_r)$ and let $S = R \cup R^{-1}$. Let β be the collection of all such S . Let $B \in \beta$ and we may as well assume $B \subseteq S$. Then there exists x such that $x = 1 - 1/(m+1)$, m a positive integer, and such that u_x , the neighborhood of x , is strictly contained in $B[1]$, the neighborhood of 1, since any neighborhood of 1 contains a tail of the sequence $\langle 1 - 1/n \rangle_{n=1}^\infty$. Then, $(x, 1) \notin x \times u_x$ but $(x, 1) \in B[1] \times 1$. In order to have a point semiuniformity, we need $D \circ (x, 1) \subseteq S$ for some $D \in \beta$. The composition

$$D \circ (x, 1) = (\{ U_{y \in \mathbb{R}} y \times D[y] \} \cup \{ U_{y \in \mathbb{R}} D[y] \times y \}) \circ (x, 1) \\ = (x \times D[1]) \cup \{ (x, y) \mid 1 \in D[y] \}$$

and recall that $S = (U(r \times u_r)) \cup (U(u_r \times r))$. Now if $x \times D[1] \subseteq \bigcup_{r \in R} U_r \times r$ then $\forall z \in D[1]$, $x \in u_z$. But $x = 1 - 1/m' \in D[1]$ for some integer $m' > m$ and $x \notin u_z$. Also, $(x, x) \notin x \times u_x$. Hence, although $(x, x) \in x \times D[1] \subseteq D \circ (x, 1)$, $(x, x) \notin (\bigcup_{r \in R} (r \times u_r)) \cup (\bigcup_{r \in R} (u_r \times r))$. Thus, $D \circ (x, 1) \not\subseteq S$.

This example shows that Page's construction of a t -semiuniformity may not be a point semiuniformity. Although t -semiuniformities and point semiuniformities are not the same, they are related as the next theorem shows.

THEOREM 4. A topological space (X, τ) is point semiuniformizable if and only if (X, τ) is p -regular.

PROOF. Since one direction is trivial, we only need to show that a p -regular space is point semiuniformizable.

Thus, let β be the collection of all neighborhoods of Δ in $\tau \times \tau$. Clearly, β is a filter that satisfies property 1) and 2) of the definition of a point semiuniformity. Therefore, to show property 3), let $B \in \beta$ which implies that there exists $U \in \tau \times \tau$ with $\Delta \subseteq U \subseteq B$. Let $C = U \cap U^{-1}$. Then C is open and symmetric and $\Delta \subseteq C \subseteq B$. Pick $(x, y) \in C$. We need to find $D \in \beta$ with $D \circ (x, y) \subseteq B$ or equivalently, $D[y] \subseteq C[x] \subseteq B[x]$ (The proof of $(x, y) \circ D \subseteq B$ is similar using inverse notation). Let $D = C - \{y \times (X - C[x])\}$.

a) To show that $D[y] \subseteq C[x]$, suppose that $z \notin C[x]$. Then $(y, z) \in y \times (X - C[x])$ and so $(y, z) \notin D$. Thus, $z \notin D[y]$.

b) To show that D is a neighborhood of Δ , consider $D' = C - \{Cl_\tau\{y\} \times (X - C[x])\}$.

Since $y \times (X - C[x]) \subseteq Cl_\tau\{y\} \times (X - C[x])$, we have that $D' \subseteq D$. Thus, since D' is open, we only need to show that $\Delta \subseteq D'$.

Case 1] Suppose $(z, z) \in \Delta$ with $z \in Cl_\tau\{y\}$. Since (X, τ) is p -regular, $y \in Cl_\tau\{z\}$ and since $(x, y) \in C$ implies that $y \in C[x]$, then $z \in C[x]$. Thus, $(z, z) \in C - \{Cl_\tau\{y\} \times (X - C[x])\} = D'$.

Case 2] Suppose $(z, z) \in \Delta$ with $z \notin Cl_\tau\{y\}$. Then $(z, z) \in D'$ and hence, $\Delta \subseteq D'$.

Clearly, $B[x] \in \tau$, $\forall B \in \beta$ and $B \in \tau \times \tau$ which implies that the topology on X generated by β is contained in τ . Conversely, let $x \in W \in \tau$. Choose a cover of X by τ -open neighborhoods $\{V_y \mid y \in X\}$ such that $V_x \subseteq W$ and $x \in X - V_y$ for $y \notin Cl_\tau\{x\}$ and $V_y = V_x$ if $y \in Cl_\tau\{x\}$ (or equivalently, $x \in Cl_\tau\{y\}$). Let $S = \bigcup_{y \in X} V_y \times V_y$. Clearly, $\Delta \subseteq S$ and S is open in $X \times X$. Also, $S[x] = \bigcup_{x \in V_y} V_y = V_x$. Hence, $x \in S[x] = V_x \subseteq W$. Therefore, W is open in the topology on X generated by β . Thus, τ is contained in the topology on X generated by β . ■

Although we used neighborhoods of Δ in the product topology in the proof above, we could have used neighborhoods of Δ in $(\tau \times \text{discrete}) \cap (\text{discrete} \times \tau)$ which would give us a finer point semiuniformity inducing the same topology..

Theorem 4 proves that t -semiuniformizable and point semiuniformizable are equivalent notions for a topological space;

however, this is their only similarity. From the examples we note that vicinities in a t -semiuniform base are constructed simply by forming "crosses" along the diagonal; whereas, in a point semiuniform base vicinities are more carefully constructed possessing "crosses" at each point.

PROPOSITION 5. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of point semiuniformities for X . Let β be the collection of all finite intersections of elements of $\bigcup_{\alpha \in \Lambda} U_\alpha$. β is a base for a point semiuniformity which is the join of the family $\{U_\alpha\}_{\alpha \in \Lambda}$.

Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be a family of point semiuniformities for the set X . Let C be the family of all point semiuniformities coarser than each V_α , $\alpha \in \Lambda$. C is a nonempty collection since $\{X \times X\}$ is a point semiuniformity coarser than each V_α , $\alpha \in \Lambda$. Then the meet of the family $\{V_\alpha\}_{\alpha \in \Lambda}$ is $\bigvee_{U \in C} U$.

Thus, if we let X be a fixed set and we consider the collection of point semiuniformities for X , then this collection forms a complete lattice when ordered by set inclusion.

THEOREM 6. If β is a finite base for a point semiuniformity then β is a base for a uniformity.

PROOF. Let $B \in \beta$. Since β is a base for a point semiuniformity, then there exists $C \in \beta$ such that for each $(x, y) \in C$ there exists $D_{(x, y)} \in \beta$ with $D_{(x, y)} \circ (x, y) \subseteq B$. Now, since β is a finite base, then the set $S = \{ D_{(x, y)} \mid (x, y) \in C \text{ and } D_{(x, y)} \circ (x, y) \subseteq B \}$ is finite. Thus, there exists $E \in \beta$ with $E \subseteq (\cap S) \cap C$. Let $(s, t) \in E \circ E$. Then there exists w such that $(s, w) \in E$ and $(w, t) \in E$. Since $E \subseteq (\cap S) \cap C$, which implies that $(s, w) \in C$, then $(w, t) \in D_{(s, w)}$. Therefore, $(s, t) = (w, t) \circ (s, w) \in D_{(s, w)} \circ (s, w) \subseteq B$. ■

COROLLARY 7. On a finite set, point semiuniformities and uniformities coincide.

3. Semicontinuous Groups

It is well known that group topologies on a group G are characterized by fundamental systems and fundamental systems give rise to left and right uniformities which give the same topology. A semifundamental system S for a group G is a collection of subsets of G each containing the identity and satisfying the following properties:

- 1) If $U, V \in S$ then $\exists W \in S$ such that $W \subseteq U \cap V$
- 2) If $U \in S$ and $a \in U$ then $\exists V \in S$ such that $Va \subseteq U$
- 3) If $U \in S$ then $\exists V \in S$ such that $V^{-1} \subseteq U$
- 4) If $U \in S$ and $x \in G$ then $\exists V \in S$ such that $xVx^{-1} \subseteq U$.

E. Clay [3] showed that every semicontinuous group has a semifundamental system. Let U be an element of a semifundamental system for a semicontinuous group. Since inversion is continuous, we

can always find a symmetric V such that $V \subseteq U$ by letting $V = U \cap U^{-1}$. Consequently, we can always assume that our semifundamental system is a symmetric semifundamental system. The next theorem shows that every semicontinuous group is point semiuniformizable.

THEOREM 8. Let $\mathcal{S}$ be a symmetric semifundamental system for a semicontinuous group (G, τ) . Define $L_0 = \{(x, y) \mid x \in yU\}$ [$R_0 = \{(x, y) \mid x \in Uy\}$]. Then $L = \{L_U \mid U \in \mathcal{S}\}$ [$R = \{R_U \mid U \in \mathcal{S}\}$] is a base for a point semiuniformity which induces the original topology τ . The point semiuniformity generated by L [R] is called the left [right] point semiuniformity of (G, τ) and is the unique point semiuniformity for G that generates τ and has a base of left [right] invariant sets.

PROOF. Clearly, L is a filter base that satisfies the first two properties of a base for a point semiuniformity.

Let $L_U \in L$ and $(x, y) \in L_U$. Then $y^{-1}x \in U$. By definition of semifundamental system, we can find $W \in \mathcal{S}$ such that $Wy^{-1}x \subseteq U$. Let $(x, z) \in L_W \circ (x, y)$. Then $(y, z) \in L_W$ which implies that $z^{-1}y \in W$. Therefore, $z^{-1}yy^{-1}x \in Wy^{-1}x \subseteq U$. Thus, $z^{-1}x \in U$ or equivalently, $(x, z) \in L_U$.

Let $L_U \in L$ and $(x, y) \in L_U$. Then $y^{-1}x \in U$. By definition of semifundamental system, we can find $W \in \mathcal{S}$ such that $Wy^{-1}x \subseteq U$. Also, there exists $V \in \mathcal{S}$ such that $y^{-1}xV(y^{-1}x)^{-1} \subseteq W$. So then $y^{-1}xV \subseteq Wy^{-1}x \subseteq U$. Let $(z, y) \in (x, y) \circ L_V$. Then $(z, x) \in L_V$ which implies that $x^{-1}z \in V$. Therefore, $y^{-1}xx^{-1}z \in y^{-1}xV \subseteq U$. Thus, $y^{-1}z \in U$, or equivalently, $(z, y) \in L_U$.

To show that this left point semiuniformity generates the topology τ , let $U \in \mathcal{S}$ and $x \in G$. Then $L_U[x] = \{y \mid (x, y) \in L_U\} = \{y \mid x \in yU\} = \{y \mid y \in xU\} = xU$. ■

THEOREM 9. Suppose that $\mathcal{S}$ is a semifundamental system for a group $(G, \cdot, \tau)$. If the left or right point semiuniformity is a uniformity, then $\mathcal{S}$ is a fundamental system.

PROOF. Assuming that $\mathcal{S}$ generates a base for the left uniformity L , then picking $U \in \mathcal{S}$ implies that $L_U \in L$ and so, by definition of L , $\exists V \in \mathcal{S}$ such that $L_V \circ L_V \subseteq L_U$. If $x \in V \cdot V$ then $x = a \cdot b$ where $a \in V$ and $b \in V$. Clearly, $x \in a \cdot V$ and $a \in e \cdot V = V$. Therefore, $(x, a) \in L_V$ and $(a, e) \in L_V$. Combining the above yields $(x, e) \in L_V \circ L_V \subseteq L_U$. Thus, $x \in e \cdot U = U$. ■

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