COMPARISON AND OSCILLATION RESULTS FOR DELAY DIFFERENCE EQUATIONS WITH OSCILLATING COEFFICIENTS

WEIPING YAN and JURANG YAN

Department of Mathematics
Shanxi University
Taiyuan, Shanxi 030006
People’s Republic of China

(Received March 25, 1993 and in revised form November 28, 1993)

ABSTRACT. In this paper we consider the oscillation of the delay difference equation with oscillating coefficients

\[ x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)x_{n-i} = 0, \quad n \geq 0. \]

Some comparison and oscillation results are obtained.

KEY WORDS AND PHRASES. Oscillation, delay difference equation, oscillating coefficient

1991 AMS SUBJECT CLASSIFICATION CODE. 39A12.

1. INTRODUCTION.

Let \( \mathbb{R} = (-\infty, \infty) \) and \( \mathbb{Z} = \{0, 1, 2, \ldots\} \). Consider the delay difference equation

\[ x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)x_{n-i} = 0, \quad n \geq 0, \tag{1.1} \]

and the delay difference inequality

\[ x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)x_{n-i} \leq 0, \quad n \geq 0, \tag{1.2} \]

where

\[ k_i(n) \in \mathbb{Z} \quad \text{for } n \in \mathbb{Z}, \tag{1.3} \]

\[ p_i(n) \in \mathbb{R} \quad \text{for } n \in \mathbb{Z}, \]

and there exist positive \( k_1, k_m \in \mathbb{Z} \), such that

\[ k_1 \geq k_1(n) \geq k_2(n) \geq \cdots \geq k_m(n) \geq k_m > 0, \tag{1.4} \]

and the following condition (A) is satisfied for \( k_1 \).

\[ (A) \begin{cases} 
(i) \quad p_1(n) \in \mathbb{R}^+, p_2(n) \in \mathbb{R}^+, \ldots, \sum_{i=1}^{m} p_i(n) \in \mathbb{R}^+ = [0, \infty); \\
(ii) \quad \text{For any } N \in \mathbb{Z}, \text{there exists } N_1 \in \mathbb{Z} \text{ such that } p_i(n) \in \mathbb{R}^+ \\
\quad \text{for any } n \in [N_1, N_1 + k_1], \text{where } i = 1, 2, \ldots, m, 
\end{cases} \]

where \([N_1, N_1 + k_1] = \{N_1, N_1 + 1, \ldots, N_1 + k_1\}\).

Let \( n_0 - k = \inf_{n \in \mathbb{Z}}(n - k_i(n)) \) and \( n_0 \geq 0 \). By a solution of (1.1) (or (1.2)) we mean a sequence \( \{x_n\} \) which is defined for \( n \geq n_0 - k \) and satisfies (1.1) (or (2.2)) for \( n \geq n_0 \).

With Eq. (1.1) and with a given “initial point” \( n_0 \geq 0 \) and “initial condition” \( a_{n_0} = k, a_n \)
Eq. (1.1) has a unique solution \( \{x_n\} \) which satisfies
\[
x_j = a_j \quad \text{for} \quad j = n_0 - k, n_0 - k + 1, \ldots, n_0.
\]

A solution \( \{x_n\} \) of Eq. (1.1) is said to be oscillatory if the terms \( x_n \) of the sequence are not eventually positive or eventually negative. Otherwise, the solution is called nonoscillatory. Eq. (1.1) is called oscillatory if every solution of the equation oscillates.

A solution \( \{x_n\} \) of Eq. (1.1) through an initial point \( n_0 \) is said to be positive if the terms \( x_n \) of the solution \( \{x_n\} \) are positive for all \( n \geq n_0 - k \).

Recently there has been a lot of interest in the oscillations of delay difference equations. See, for example, \([1] - [5]\) and the references cited therein. Our aim in this paper is to study the oscillation of Eq. (1.1). Some necessary and sufficient conditions and some easily verifiable sufficient conditions are established for oscillation of Eq. (1.1).

2. MAIN RESULTS.

Consider a sequence \( \{A_n^{(r)}\}_{r=0}^{\infty} = 0 \), which is defined as
\[
A_{n}^{(r)} = \begin{cases} 
0 & \text{for } n = n_0 - k, n_0 - k + 1, \ldots, n_0 - 1, \\
\prod_{j=n-k(n)}^{n-1} (1 - A_{j-1}^{(r-1)})^{-1} & \text{for } n \geq n_0.
\end{cases}
\]

(2.1)

First, we introduce the following Lemmas.

**Lemma 1.** Assume that condition (A) holds for \( k_1 \) and \( \{x_n\} \) is an eventually positive solution of (1.1). Then, \( \{x_n\} \) must be eventually nonincreasing. And, we have
\[
x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)x_{n-k_i(n)} \leq 0.
\]

(2.2)

**Proof.** By condition (A), there exists \( N_1 \geq n_0 \) such that
\[
x_{n+1} - x_n \leq x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)x_{n-k_i(n)} = 0, \quad \text{for } n \in [N_1, N_1 + k_1],
\]

that is, \( \{x_n\} \) is nonincreasing on \([N_1, N_1 + k_1]\).

We claim that \( \{x_n\} \) is nonincreasing for \( n \in [N_1 + k_1, N_1 + k_1 + k_m] \).

In fact, for any \( n \in [N_1 + k_1, N_1 + k_1 + k_m] \), we have \( n - k_l(n) \in [N_1, N_1 + k_1] \).

By (1.4) and nonincreasing property of \( \{x_n\} \) on \([N_1, N_1 + k_1]\), we have that
\[
x_{n-k_1} \geq x_{n-k_1(n)} \geq \cdots \geq x_{n-k_2(n)} \geq x_{n-k_2} > 0,
\]

(2.3)

for any \( n \in [N_1 + k_1, N_1 + k_1 + k_m] \).

So, we get that
\[
x_{n+1} - x_n = - \sum_{i=1}^{m} p_i(n)x_{n-k_i(n)}
= - p_1(n)x_{n-k_1(n)} - \sum_{i=1}^{m} p_i(n)x_{n-k_i(n)}
\]
Therefore, \( x_n \) is nonincreasing on \([N_1+k_1, N_1+k_1+k_m]\). Similarly, we can show that \( \{x_n\} \) is nonincreasing for all \( n \geq N_1+k_1 \) and the proof is complete.

**Lemma 2.** Assume that condition (A) holds for \( k_1 \) and (1.1) has a positive solution. Then there exists a sequence \( \{a_n\}_{n=n_0-k_1} \) such that the following statements are true:

1. \( a_n = \sum_{j=1}^{m} p_j(n) \prod_{j=n-k_1(n)}^{n-1} (1-a_j)^{-1}, \quad \text{for } n \geq n_0; \)
2. \( a_n < 1 \) for \( n = n_0 - k_1, n_0 - k_1 + 1, \ldots, n_0 - 1 \) and eventually \( 0 \leq a_n < 1 \) for \( n \geq N_1 \).

**Proof.** Assume that \( \{x_n\} \) is a solution of (1.1) and \( x_n > 0 \) for all \( n \geq n_0 - k_1 \). Set

\[
a_n = 1 - \frac{x_{n+1}}{x_n}, \quad \text{for } n \geq n_0 - k. \tag{2.4}
\]

Then

\[
\frac{x_{n-k_1(n)}}{x_n} = \frac{x_{n-k_1(n)} \cdot x_{n-k_1(n)+1} \cdot \ldots \cdot x_{n-1}}{x_n}
\]

\[
= \prod_{j=n-k_1(n)}^{n-1} (1-a_j)^{-1}, \quad n \geq n_0. \tag{2.5}
\]

From (1.1), we have that

\[
\frac{x_{n+1}}{x_n} - 1 + \sum_{j=1}^{m} p_j(n) \frac{x_{n-k_1(n)}}{x_n} = 0, \quad n \geq n_0. \tag{2.6}
\]

Hence, by substituting (2.4) and (2.5) into (2.6), we get

\[
a_n = \sum_{j=1}^{m} p_j(n) \prod_{j=n-k_1(n)}^{n-1} (1-a_j)^{-1}, \quad n \geq n_0, \tag{2.7}
\]

that is, (i) is satisfied. Clearly, \( a_n < 1 \) for \( n \geq n_0 - k_1 \). By Lemma 1, we have eventually \( 0 \leq a_n < 1 \). The proof of Lemma 2 is completed.

**Lemma 3.** Assume that condition (A) holds for \( k_1 \) and (1.1) has a positive solution through \( n_0 \). Then the sequence \( \{a_n\} \) is well defined for \( n \geq n_0 \) and satisfies

1. \( 0 \leq A_n^{(r)} A_n^{(r+1)} , \quad \text{for } n \geq n_0 \) and \( r \geq 0; \)
2. \( \lim_{r \to \infty} A_n^{(r)} \) def. \( A_n < 1 \), \quad \text{for } n \geq n_0.

**Proof.** Assume that \( \{x_n\} \) is a positive solution of (1.1) through \( n_0 \). By Lemma 1, without loss of generality, we assume \( \{x_n\} \) nonincreasing as \( n \geq n_0 - k_1 \).

Set \( a_n = 1 - \frac{x_{n+1}}{x_n} \) for \( n \geq n_0 - k_1 \). Then from (2.4) and Lemma 1, and by a simple induction, it can be seen that
In fact, $A_{(n)}^{(d)} = 0$, and

$$A_{(n)}^{(d)} = \begin{cases} 0, & \text{for } n = n_0 - k_1, n_0 - k_1 + 1, \ldots, n_0 - 1. \\
\sum_{i=1}^{n} \rho_i(n), & \text{for } n \geq n_0. 
\end{cases}$$

So, we have $A_{(n)}^{(d)} \geq A_{(n)}^{(d)} \geq 0$. Assume that $A_{(n)}^{(r)} \geq A_{(n)}^{(r-1)} \geq 0$. Then, for $n \geq n_0$, we have

$$A_{(n)}^{(r+1)} = \sum_{i=1}^{n} \rho_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - A_j^{(r)})^{-1},$$

$$A_{(n)}^{(r)} = \sum_{i=1}^{n} \rho_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - A_j^{(r-1)})^{-1},$$

Hence, we get

$$A_{(n)}^{(r+1)} - A_{(n)}^{(r)} = \sum_{i=1}^{n} \rho_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - A_j^{(r-1)})^{-1} \left[ \prod_{j=n-k_i(n)}^{n-1} \frac{(1 - A_j^{(r-1)})}{(1 - A_j^{(r)})} - 1 \right]$$

$$\geq \left[ \sum_{i=1}^{n} \rho_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - A_j^{(r-1)})^{-1} \left[ \prod_{j=n-k_i(n)}^{n-1} \frac{(1 - A_j^{(r-1)})}{(1 - A_j^{(r)})} - 1 \right] \right] \geq 0,$$

so, we know that $0 \leq A_{(n)}^{(r)} \leq A_{(n)}^{(r+1)}$ for all $r \geq 0$.

By (2.7), we use the induction to get

$$A_{(n)}^{(r)} \leq a_r < 1, \quad \text{for any } r \geq 0, n \geq n_0 - k_1,$$

which implies that (2.8) holds. Hence it is easy to get that

$$\lim_{n \to \infty} A_{(n)}^{(r)} = a_r < 1, \quad n \geq n_0,$$

and the proof is complete.

The next result is a generalization of Theorem 1 in [5].

**THEOREM 1.** Assume that (1.3), (1.4) and condition (A) hold for $k_1$. Then the following statements are equivalent:

(a) Eq. (1.1) has a positive solution through the initial point $n_0 \geq 0$;

(b) The inequality (1.2) has an eventually positive solution;

(c) The sequence $(A_{(n)}^{(r)})_{r=0}^{\infty}$ which is well defined by (2.1) converges to a limit $A_{(n)}$ with $0 \leq A_{(n)} < 1$ for each $n \geq n_0 > 0$.

**PROOF.** (a)$\Rightarrow$(b). This is obvious.

(b)$\Rightarrow$(c). Assume that $x_n > 0$ for $n \geq n_0 - k$ which is a solution of (1.2). Set

$$\tilde{a}_s = 1 - \frac{x_{s+1}}{x_s}, \quad n \geq n_0 - k.$$

Then,

$$\frac{x_{n-k_i(n)}}{x_n} = \prod_{j=n-k_i(n)}^{n-1} (1 - \tilde{a}_j)^{-1}, \quad n \geq n_0. \quad (2.9)$$

Thus from (1.2), it follows that

$$\sum_{i=1}^{n} \rho_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - a_j)^{-1} \leq \tilde{a}_s, \quad n \geq n_0.$$

By (2.1) and a simple induction which is the same as that of Lemma 3, we have that
0 \leq A^{(r)}_n \leq A^{(r+1)}_n \leq a < 1 \quad \text{for } r \geq 0 \text{ and } n \geq n_0,

which implies that the sequence \( \{A^{(r)}_n\} \) converges to finite limit \( A_n \) with \( 0 \leq A_n < 1 \) for each fixed \( n \geq n_0 \).

(c) \Rightarrow (a). It is similar to that of Theorem 1 in [5].

The proof of Theorem 1 is complete.

COROLLARY 1. Assume that (1.3), (1.4) and condition (A) hold for \( k_1 \).

Then the following statements are equivalent:

(a) Eq. (1.1) is oscillating;

(b) Inequality (1.2) has no eventually positive solution.

Assume that \( P_n \in \mathbb{R}^+, n \in \mathbb{Z}, \) and \( k \in \mathbb{Z}. \) The following theorem of oscillation was obtained in [3].

THEOREM A. Consider the delay difference equation

\[ A_{n+1} - A_n + P_n A_{n-k} = 0, \quad n = 0,1,2,\ldots. \quad (\ast) \]

If

\[ \liminf_{n \to \infty} \left( \frac{1}{k} \sum_{i=n-k}^{n-1} P_i \right) > \frac{k^k}{(k+1)^{k+1}}, \quad (2.10) \]

then all solutions of (\ast) are oscillatory.

In [5], the following conclusion was obtained.

THEOREM B. Consider the following inequality

\[ A_{n+1} - A_n + P_n A_{n-k} \leq 0. \quad (\ast \ast) \]

The following conclusions are equivalent:

(i) (\ast \ast) is oscillatory;

(ii) (\ast \ast) has no eventually positive solution.

We can obtain the following theorem.

THEOREM 2. Assume that (1.3), (1.4) and condition (A) hold for \( k_1, \) and the equation

\[ x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i} = 0 \quad (2.11) \]

is oscillatory, then (1.1) must be oscillatory.

PROOF. Let \( \{x_n\} \) be a nonoscillatory solution of (1.1). As the opposite of a solution of (1.1) is still a solution of (1.1), we can assume that \( x_n > 0 \) for \( n \geq n_0. \) By Lemma 1, we have

\[ x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i} \leq 0, \quad (2.12) \]

that is, inequality (2.12) has a positive solution. On the other hand, by Theorem B we know that (2.12) has no eventually positive solution. This is a contradiction. So, (1.1) must be oscillatory. The proof of Theorem 2 is complete.

COROLLARY 2. Assume that (1.1), (1.4), condition (A) hold for \( k_1, \) and

\[ \liminf_{n \to \infty} \left( \frac{1}{k} \sum_{i=n-k}^{n-1} \sum_{j=1}^{m} p_{i,j} \right) > \frac{k^k}{(k+1)^{k+1}}. \quad (2.13) \]

Then (1.1) is oscillatory.
PROOF. By Theorem 2 and Theorem A, we obtain the conclusion.

COROLLARY 3. Assume that (1.3), (1.4), condition(A) hold for \( k, \) and

\[
\sum_{i=1}^{n} p_i(n) \geq P_n \quad \text{for } n \text{ sufficiently large.} \tag{2.14}
\]

Then, if \((*)\) is oscillatory, (1.1) must be oscillatory.

REFERENCES

Submit your manuscripts at http://www.hindawi.com