ON NON-PARALLEL s-STRUCTURES

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ABSTRACT. Using algebraic topology, we find out the number of all non-parallel s-structures which an n-dimensional Euclidean space \( E^n \) admits. The obtaining results are generalized on a manifold \( M \) which is CW-complex.

KEY WORDS AND PHRASES. Non-parallel s-structure, CW-complex

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0. INTRODUCTION

Let \( E^n \) be an n-dimensional Euclidean space and \( f : E^n \rightarrow E^n \) an automorphism.

If \( x_0 \in E^n \) then the expression \( f(x_0) + Df_{x_0}(x - x_0) \) is the linear approximation of \( f \) at \( x_0 \).

We assume that \( x_0 \) is a fixed point of \( f \) and the Jacobian matrix \( Df \) is an orthogonal matrix. Then, if in a closed neighborhood of \( x_0 \) (under the usual topology) there is no other fixed point, \( f \) is called s-symmetry on \( x_0 \) and it is written

\[
L_{x_0}(x) = x_0 + A_{x_0}(x - x_0),
\]

where the Jacobian \( A_{x_0} \) belongs to \( O(n) - \{I\} \) (if \( A_{x_0} = I \), then every point of \( E^n \) will be a fixed point).

A family \( \{f_{x_0} : x_0 \in E^n\} \) of s-symmetries is called an s-structure on \( E^n \).

An s-structure is called regular if \( f_{x_0} \cdot f_{y_0} = f_{u_0} \cdot f_{x_0} \), where \( u_0 = f_{x_0}(y_0) \).

An s-structure is called parallel if \( A_{x_0} \) is constant i.e., it does not depend on \( x_0 \). It is clear that a parallel s-structure is also a regular one. If \( f_{0} \) is an orthogonal transformation at the origin without fixed vectors and \( t_{x_0} \) is a translation on \( E^n \) such that \( t_{x_0}(0) = x_0 \) then

\[
f_{x_0} = t_{x_0} \circ f_{0} \circ t_{x_0}^{-1}
\]

are the only parallel s-structures on \( E^n \).

Therefore the following question arises: Do there exist non-parallel regular s-structures on \( E^n \)?

O. Kowalski in [1] proved that the Euclidean spaces \( E^2, E^3 \) and \( E^4 \) admit only parallel regular s-structures and found out a non-parallel regular s-structure on \( E^5 \).

S. Wegrzynowski in [2] obtained the same results using analytical calculations on Lie algebras.

In the present paper, we will give a complete classification of the Euclidean spaces of arbitrary dimension admitting non-parallel s-structures and we will give the number of these ones as well. Finally we will generalize the meaning of an s-structure on every manifold which is a CW-complex and we will solve the analogous problem on these manifolds. We will prove that the number of the non-parallel s-structures is a dimensional-invariant.
1. Euclidean Spaces

In the present paragraph we will prove the following

**THEOREM 1.** In an \( n \)-dimensional Euclidean space \( E^n \) the number \( N \) of the non-parallel \( s \)-structures is given by

\[
N = \begin{cases} 
0 & \text{for } n = 2, 3, 4, \\
2^{n-1}(2^n - 1) & \text{for } n \geq 5.
\end{cases}
\]

**PROOF.** Let \( f_z \) be an \( s \)-symmetry on \( E^n \), i.e., an isometry with an isolated fixed point \( x_0 \)

If \( x_0 \) and \( x'_0 \) are two fixed points, then we can find positive numbers \( \epsilon \) and \( \epsilon' \) such that (under the usual topology in \( E^n \)) \( x_0 \notin \overline{N}(x'_0, \epsilon') \) and \( x'_0 \notin \overline{N}(x_0, \epsilon) \), where \( \overline{N}(x_0, \epsilon) (\overline{N}(x'_0, \epsilon')) \) is the closure of the open neighborhood of \( x_0 (x'_0) \) with radius \( \epsilon (\epsilon') \)

So, we can substitute the fixed point \( x_0 \) with the neighborhood \( \overline{N}(x_0, \epsilon) \) preserving the geometrical properties of \( f_z \). Then, \( f_z \) becomes

\[
f_z : E^n - \overline{N}(x_0, \epsilon) \rightarrow E^n - \overline{N}(x_0, \epsilon),
\]

where \( \overline{N}(x_0, \epsilon) \) is invariant under the action of \( f_z \).

Denoting \( \tilde{E}_z(\epsilon) = E^n - \overline{N}(x_0, \epsilon) \), \( f_z \) takes the form

\[
f_z : \tilde{E}_z(\epsilon) \rightarrow \tilde{E}_z(\epsilon).
\]

Using an orthogonal coordinate system in \( E^n \) we have

\[
\tilde{E}_z(\epsilon) = \left\{ (x_1 - x_1^0, x_2 - x_2^0, ..., x_n - x_n^0) / \sum_{i=1}^{n} (x_i - x_i^0)^2 > \epsilon \right\}.
\]

If we define

\[
\tilde{E}_{z_0}^{n-1}(\epsilon, j) = \left\{ (x_1 - x_1^0, ..., x_{j-1} - x_{j-1}^0, x_j + x_j^0, ..., x_n - x_n^0) / \sum_{i=1, i \neq j}^{n} (x_i - x_i^0)^2 > \epsilon \right\}
\]

then \( \tilde{E}_z(\epsilon) \) can be decomposed to the "direct sum" of the above \( (n - 1) \)-dimensional subspaces as

\[
\tilde{E}_z(\epsilon) = \bigoplus_{j=1}^{n-1} \tilde{E}_{z_0}^{n-1}(\epsilon, j) \quad \text{def.}
\]

\[
= \left\{ \frac{1}{n-1} \left[ (0, x_2 - x_2^0, ..., x_n - x_n^0)^T + ... + (x_1 - x_1^0, x_2 - x_2^0, ..., 0)^T \right] / \sum_{j=1}^{n} (x_j - x_j^0)^2 > \epsilon \right\}.
\]

The action of an orthogonal matrix \( A_z \) on \( \tilde{E}_{z_0}^{n-1}(\epsilon, j) \) has the form

\[
A_z \begin{pmatrix} x_1 - x_1^0 \\ \vdots \\ x_n - x_n^0 \end{pmatrix} = \frac{1}{n-1} A_z \begin{pmatrix} 0 \\ \vdots \\ x_n - x_n^0 \end{pmatrix} + ... + \begin{pmatrix} x_1 - x_1^0 \\ \vdots \\ 0 \end{pmatrix}.
\]

We observe that this action can be decomposed to a sum of mutually independent parts

Choosing the \( j \)-th part where \( 1 \leq j \leq n \), we shall prove that if \( x_0, x'_0 \) are two different fixed points in \( E^n \) we can pass from \( \tilde{E}_{z_0}^{n-1}(\epsilon, j) \) to \( \tilde{E}_{z_0}^{n-1}(\epsilon', j) \) for every \( j \), where

\[
\tilde{E}_{z_0}^{n-1}(\epsilon', j) = A_{z_0}(\tilde{E}_{z_0}^{n-1}(\epsilon', j)).
\]

Taking \( \tilde{E}_{z_0}^{n-1}(\epsilon*, j) \) with \( \epsilon* = \min\{\epsilon, \epsilon'\} \) we have the following commutative diagram
where, $\alpha_j : S^{n-1}_x(e^*) \xrightarrow{st} E^{n-1} \xrightarrow{h_4} E^{n-2} \times E^1 \xrightarrow{q} E^n-2 \times S^1 \xrightarrow{h_5} \tilde{E}^{n-1}_{x_0}(e^*, j)$, and $h_4$ are homeomorphisms, $q_1 \times I$ is the natural map, $q_2$ is the quotient map, $q = (id, q_{S^1})$ is the quotient map and $S^{n-1}_x(e^*)$ is the $(n-1)$-sphere with center $x_0$ and radius $e^*$ under quotient topology ($V \subset S^{n-1}$ is open if and only if $q^{-1}(V)$ is open)

Repeating the above diagram for every $j$, it turns out that the existence of $\phi_j$ depends on the existence of $\alpha_j$, which are classified by definition from the $n-1$ homotopy group of $\tilde{E}^{n-1}_{x_0}(e, j)$. But $\tilde{E}^{n-1}_{x_0}(e, j)$ is of the same homotopy type with $S^{n-1}_x(e)$, hence $\pi_{n-1}(\tilde{E}^{n-1}_{x_0}(e, j)) \cong \pi_{n-1}(S^{n-2})$.

Finally, we obtain that the mapping

$$
\bigoplus_{j=1}^{n} \tilde{E}^{n-1}_{x_0}(e, j) \to \bigoplus_{j=1}^{n} \tilde{E}^{n-1}_{x_0}(e, j)
$$

exists if the corresponding maps $\alpha_j$ belong to the same homotopy equivalence class.

It is well known that

- for $n = 2$, $\pi_1(S^0) = 0$,
- for $n = 3$, $\pi_2(S^1) = 0$,
- for $n = 4$, $\pi_3(S^2) = \mathbb{Z}$.

Hence, it is clear that the spaces $E^2$, $E^3$ and $E^4$ admit only parallel $s$-structures because $\alpha_j$'s exist and belong to the same homotopy equivalence class.

For $n \geq 5$ we have $\pi_{n-1}(S^{n-2}) = \mathbb{Z}_2$, so the spaces $E^n$ for $n \geq 5$ admit non-parallel $s$-structures.

To find out the number of the non-parallel $s$-structures of a Euclidean space $E^n (n \geq 5)$ we consider the case $n = 5$

Composing again the 4-dimensional spaces we have

$$\tilde{E}^5_{x_0}(e) = \tilde{E}^4_{x_0}(e, 1) \oplus \cdots \oplus \tilde{E}^4_{x_0}(e, 5),$$

and

$$\star \tilde{E}^5_{x_0}(e) = \star \tilde{E}^4_{x_0}(e, 1) \oplus \cdots \oplus \star \tilde{E}^4_{x_0}(e, 5).$$

If 0 and 1 are the classes of $\mathbb{Z}_2$ then every $\tilde{E}^4_{x_0}(e, j)$ and $\star \tilde{E}^4_{x_0}(e, j)$ corresponds to 0 or 1. Hence

$$\tilde{E}^5_{x_0} = \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4 \oplus \alpha_5,$$

and
The passing from \( \tilde{E}_{i_j}^3 \) to \( \tilde{E}_{i_j}^5 \) can be done by a parallel way, if and only if \( \tilde{\alpha}_i = \alpha_i \) for every \( i = 1, \ldots, 5 \).

Obviously, there exist \( 2^5 = 32 \) 5-tuples and \( \binom{32}{2} = 496 \) non-parallel mappings.

The above proof we can apply to the \( n \)-dimensional Euclidean space, and so the proof of the theorem is completed.

2. CW-COMPLEXES

In the present paragraph we generalize the results of the first one.

**Theorem 2.** Let \( M \) be an \( n \)-dimensional manifold which is a CW-complex. Then, \( M \) admits \( N \) non-parallel \( s \)-structures where \( N \) is given by

\[
N = \begin{cases} 
0 & \text{if } n < 5, \\
2^{n-1}(2^n - 1) & \text{if } n \geq 5.
\end{cases}
\]

**Proof.** \( M \) is a CW-complex, hence it can be decomposed as

\[
M = e_{i_1} \sqcup e_{i_2} \sqcup \ldots \sqcup e_{i_n}, \quad i_1 \leq i_2 \leq \ldots \leq i_n,
\]

where \( e_{i_n} \) is the maximal-dimension cell and \( \dim M = \dim e_{i_n} \).

We have to take the fixed point on the cell \( e_{i_n} \). Otherwise the fixed point will not be isolated.

We consider the diagram

\[
e_{i_1} \quad \ldots \quad e_{i_{n-1}} \quad e_{i_n} \quad h_1 \quad E^n_{x_0} \\
I \quad I \quad f \\
e_{i_1} \quad \ldots \quad e_{i_{n-1}} \quad e'_{i_n} \quad h_2 \quad E^n_{x_0}
\]

where \( h_1 \) and \( h_2 \) are homeomorphisms and \( f_{x_0} \) is defined as in Theorem 1. Also, we define \( f \) to be non-parallel if and only if it does not depend on \( x_0 \).

Thus, we can study the maps \( f_{x_0} \) instead of \( f \). The last suggestion completes the proof of Theorem 2.

**Examples:**

1. \( S^6 = e_1 \sqcup e_6, \quad N = \binom{2^6}{2} = 2016, \)
2. \( CP(10) = e_0 \sqcup e_2 \sqcup e_4 \sqcup e_6 \sqcup e_8 \sqcup e_9 \sqcup e_{10}, \quad N = 2^6(2^{10} - 1) = 523,776, \)
3. \( RP(6) = e_0 \sqcup e_1 \sqcup e_2 \sqcup e_3 \sqcup e_4 \sqcup e_5 \sqcup e_6, \quad N = 2^5(2^6 - 1) = 2016. \)

Considering the first and third example of the second paragraph we observe that two manifolds which are CW-complexes \( (S^6, \mathbb{R}P(6)) \) have the same number of non-parallel \( s \)-structures although they have different geometrical and topological structures. Thus, the following questions arises: "Do there exist manifolds admitting \( N \) non-parallel \( s \)-structures where \( N \neq \binom{2^n}{n}, \quad n \geq 5 \) ?"

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**References**


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