CHAIN CONDITIONS ON SEMIRINGS

T. K. MUKHERJEE, M. K. SEN and SHAMIK GHOSH

Department of Pure Mathematics, University of Calcutta,
35, Ballygunge Circular Road, Calcutta - 700 019, India.

(Received May 4, 1994 and in revised form January 7, 1995)

ABSTRACT . In this paper we characterize the class of semirings S for which the semirings of square matrices $M_n(S)$ over S are (left) k -artinian. Also an analogue of the Hilbert basis theorem for semirings is obtained.

KEY WORDS AND PHRASES . Semiring, halfiring, k -ideal, h -ideal, artinian semiring, noetherian semiring.

1991 AMS SUBJECT CLASSIFICATION CODE . Primary 16Y60.

0. INTRODUCTION

A *semiring* S is defined as an algebraic system $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups, connected by $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. An (*absorbing*) *zero* element of a semiring S is an element 0 such that $0 + x = x + 0 = x$ and $0x = x0 = 0$ for all $x \in S$. For the rest of the paper we assume that a semiring is additively commutative and has a zero element. If moreover a semiring S is additively cancellative, then it is called a *halfiring*. A *semifield* is a semiring in which non-zero elements form a group under multiplication.

We know that the ring $M_n(R)$ of $n \times n$ matrices over a ring R is (left) artinian iff R is (left) artinian. But examples show that there are (left) artinian semirings (even semifields) S for which $M_n(S)$ is not (left) artinian.

In this paper we characterize the class of semirings S such that all $M_n(S)$ are (left) k -artinian (cf. Definition 1.1). Another characterization of the class of semirings S for which all $M_n(S)$ are (left) h -artinian is obtained.

We also obtain an analogue of the Hilbert basis theorem for semirings which generalizes a result of H. E. Stone [1].

1. CHAIN CONDITIONS ON MATRIX SEMIRINGS

Let S be a semiring. A subsemiring I of S is said to be a *left ideal* of S if $ra \in I$ for all $r \in S$ and $a \in I$. A *left k -ideal* [*left h -ideal*] is a left ideal of S for which $x \in S$ [$x, z \in S$], $a, b \in I$ and $x + a = b$ [$x + a + z = b + z$] imply $x \in I$ [2].

DEFINITION 1.1 A semiring S is said to be *(left) artinian* [*k-artinian*, *h-artinian*] if S satisfies the descending chain condition on left ideals [*k-ideals*, *h-ideals*] of S .

Obviously, artinian implies *k-artinian* and the latter implies *h-artinian*.

A *(left) semimodule* M over a semiring S is a commutative additive semigroup which has a zero element, together with a mapping from $S \times M$ into M (sending (r, m) to rm) such that

$$\begin{aligned} \text{(i)} \quad (r + s)m &= rm + sm, & \text{(ii)} \quad r(m + p) &= rm + rp, \\ \text{(iii)} \quad r(sm) &= (rs)m, & \text{(iv)} \quad 0m &= r0 = 0 \end{aligned}$$

for all $m, p \in M$ and $r, s \in S$ [3]. We define *k-subsemimodules* and *h-subsemimodules* of an S -semimodule and *k-artinian* and *h-artinian semimodule* over a semiring S in a similar fashion.

Let S be a semiring with a multiplicative identity 1 and $M_n(S)$ be the semiring of $n \times n$ matrices over S . Let E_{ij} be the matrix in $M_n(S)$ such that its (i, j) th. entry is 1 and all other entries are zero. Henceforth for any matrix $A = (a_{ij}) \in M_n(S)$ and $i = 1, 2, \dots, n$, we introduce

$$A_i = E_{ii}A = \sum_{j=1}^n a_{ij}E_{ij}.$$

Then A can be written as $A_1 + A_2 + \dots + A_n$. The i th. row matrix $(a_{i1} \ a_{i2} \ \dots \ a_{in})$ of A will be denoted by $\mathbf{a}_i$, $i = 1, 2, \dots, n$. Let I be any left ideal of $M_n(S)$. We define for each $i = 1, 2, \dots, n$, $I_{0i} = \{A_i : A \in I\}$ and $I_i = \{\mathbf{a}_i : A = (a_{ij}) \in I\}$. Now $I_{0i} \subseteq I$, as I is a left ideal of $M_n(S)$. Therefore $I = I_{01} \oplus I_{02} \oplus \dots \oplus I_{0n}$, where $\oplus$ means the internal direct sum (as in the case of a ring).

Now we verify that $I_1 = I_2 = \dots = I_n = M$ (say). Let $i \neq j$ and $\mathbf{a} \in I_j$. Let $A \in I$ be any matrix corresponding to $\mathbf{a}$. Now $E_{ij}A \in I$ and the i th. row matrix of $E_{ij}A$ is $\mathbf{a}$. Thus $\mathbf{a} \in I_i$. This implies $I_j \subseteq I_i$ for any $i \neq j$. This completes the verification.

Straightforward calculations show : If I is a left *k-ideal* [*h-ideal*] of $M_n(S)$, the same holds for all I_{0i} , and M is a *k-subsemimodule* [*h-subsemimodule*] of the S -semimodule S^n (considering elements of S^n as row matrices).

Conversely, let M be a subsemimodule [*k-subsemimodule*, *h-subsemimodule*] of the S -semimodule S^n and $I = M_{01} + M_{02} + \dots + M_{0n}$, where $M_{0i} = \{\sum_{j=1}^n a_{ij}E_{ij} \in M_n(S) : \mathbf{a}_i = (a_{i1}, a_{i2}, \dots, a_{in}) \in M\}$.

Clearly, I is closed under addition. Let $A \in I$, $C \in M_n(S)$. We have

$$CA = (C_1 + C_2 + \dots + C_n)(A_1 + A_2 + \dots + A_n) = \sum_{i=1}^n \sum_{j=1}^n C_i A_j.$$

Now $C_i A_j$ is a matrix in $M_n(S)$ whose all rows are zero except the i th. row, which is $(c_{ij}a_{j1} \ c_{ij}a_{j2} \ \dots \ c_{ij}a_{jn}) = c_{ij}\mathbf{a}_j \in M$, as M is a subsemimodule of S^n . Thus $C_i A_j \in M_{0i}$ and hence $CA \in I$. Therefore I is a left ideal of $M_n(S)$. Also it is trivial to show that I is a left *k-ideal* [*h-ideal*] of $M_n(S)$ if M is a *k-subsemimodule* [*h-subsemimodule*] of S^n . Thus we have proved the following lemma.

LEMMA 1.2. Let S be a semiring with a multiplicative identity 1. Then I is a left ideal [*k-ideal*, *h-ideal*] of $M_n(S)$ iff there exists a subsemimodule [*k-subsemimodule*, *h-subsemimodule*] M of the S -semimodule S^n such that $I = M_{01} \oplus M_{02} \oplus \dots \oplus M_{0n}$, where M_{0i} is defined as above.

COROLLARY 1.3 *Let S be a semiring with multiplicative identity 1. Then $M_n(S)$ is artinian [k -artinian, h -artinian] iff S^n is an artinian [k -artinian, h -artinian] S -semimodule.*

EXAMPLE 1.4. Let S be the set of all sequences of positive rationals and the constant sequence $(0, 0, 0, \dots)$ with pointwise addition and multiplication. Clearly S is a commutative semifield and hence artinian. Now let

$$M_r = \{ (x, y) \in S^2 : x = (x_1, x_2, \dots), y = (y_1, y_2, \dots), x_i = y_i, 1 \leq i \leq r \}.$$

Then $M_1 \supset M_2 \supset M_3 \supset \dots$ is an infinite descending chain of k -subsemimodules of the S -semimodule S^2 . Thus $M_2(S)$ is not k -artinian, which yields the same for all $M_n(S)$, $n \geq 2$.

Now we present another characterization of semirings over which semirings of matrices are h -artinian. It is easier to handle and it shows that actually the descending chain condition on h -ideals of $M_n(S)$ ($n > 1$) over a semiring S does not depend on n (cf. Corollary 1.11). We proceed through some preliminary lemmata.

LEMMA 1.5. *Let S be a semiring. Then every homomorphic image of a k -artinian S -semimodule is also k -artinian.*

PROOF. Let M be an S -semimodule and $\Psi: M \longrightarrow N$ be a homomorphism of M onto an S -semimodule N . It is well known that $\Psi^{-1}(K)$ is a k -subsemimodule of M for each k -subsemimodule K of N . This yields the assertion as in the corresponding proof in the case of rings ■

Let H be a halfring. We recall that H can be embedded into a ring and that the smallest ring of this kind is uniquely determined (upto isomorphism). Since the latter consists of all differences $a - b$ for $a, b \in H$, it is called the *difference ring* of H and is denoted by $D(H)$ [4].

LEMMA 1.6. *If M is an H -semimodule for a halfring H such that $(M, +)$ is a group, then M is also a $D(H)$ -module under the definition $(r_1 - r_2)m = r_1m - r_2m$ for all $r_i \in H$ and $m \in M$. Moreover any k -subsemimodule of M is a submodule of the $D(H)$ -module M and conversely.*

PROOF. One can easily check that $(r_1 - r_2)m$ is well defined and satisfies (i) to (iv) of the definition of semimodules. Let K be a k -subsemimodule of M and $m_1, m_2 \in K$. Then $(m_1 - m_2) + m_2 = m_1$ implies $m_1 - m_2 \in K$. For $m \in K$ and $r = r_1 - r_2 \in D(H)$, from $(r_1m - r_2m) + r_2m = r_1m$, we get $rm = r_1m - r_2m \in K$. Thus K is a submodule of the $D(H)$ -module M .

Conversely, let K be a submodule of the $D(H)$ -module M . Clearly K is also a subsemimodule of the H -semimodule M . Also if $u + m_1 = m_2$ for some $u \in M$, $m_1, m_2 \in K$, then $u = m_2 - m_1 \in K$. Thus K is a k -subsemimodule of M , as required ■

LEMMA 1.7. *Let H be a halfring. If H^n is a k -artinian H -semimodule ($n > 1$), then $D(H)$ is an artinian ring.*

PROOF. We define a mapping $\Psi: H^n \longrightarrow D(H)$ by

$$\Psi((a_1, a_2, \dots, a_n)) = \begin{cases} a_1 - a_2 + a_3 - \dots + a_n, & \text{when } n \text{ is even} \\ a_1 - a_2 + a_3 - \dots - a_n, & \text{when } n \text{ is odd.} \end{cases}$$

Clearly Ψ is a well defined $(H-)$ semimodule homomorphism of H^n into $D(H)$. We show that Ψ is also surjective. Let $x \in D(H)$. Then there are $x', x'' \in H$ such that $x = x' - x''$. Thus

$\psi((x', x'', 0, 0, \dots, 0)) = x$. Therefore $D(H)$ is a k -artinian H -semimodule by Lemma 1.5, which implies $D(H)$ is an artinian $D(H)$ -module by Lemma 1.6. Hence $D(H)$ is an artinian ring ■

It is well known that the left k -ideals of a halfring H are precisely the intersection with H of the left ideals of $D(H)$ [1]. From this fact it is obvious that if $D(H)$ is artinian then H is k -artinian. But the converse is not true. For example, let us consider the semifield S described in the Example 1.4. S is an artinian halfring but $D(S)$, being the countably infinite copies of all rationals, is not artinian.

LEMMA 1.8. *Let H be a halfring. If $D(H)$ is artinian, then $M_n(H)$ is k -artinian ($n \geq 1$).*

PROOF. If $D(H)$ is artinian, then so is $M_n(D(H))$. Moreover $D(M_n(H)) = M_n(D(H))$ [5]. Hence $M_n(H)$ is k -artinian ■

The following theorem follows from Corollary 1.3, Lemma 1.7 and Lemma 1.8.

THEOREM 1.9. *Let H be a halfring with multiplicative identity 1. Then the following three statements are equivalent for $n > 1$:*

- (i) $M_n(H)$ is a k -artinian halfring.
- (ii) H^n is a k -artinian H -semimodule.
- (iii) $D(H)$ is an artinian ring.

Let S be a semiring. We know that $\bar{\Delta}_S = \{(x, y) \in S \times S : x + z = y + z \text{ for some } z \in S\}$ is the least additively cancellative congruence on S and hence $S/\bar{\Delta}_S$ is a halfring. Generalizing this concept, $D(S/\bar{\Delta}_S)$ is also called the *difference ring* of the semiring S and denoted by $\tilde{S}$. We denote the $\bar{\Delta}_S$ -class of any element $a \in S$ by $[a]$. Now straightforward calculations show that $M_n(S/\bar{\Delta}_S)$ is isomorphic to $M_n(S)/\bar{\Delta}_{M_n(S)}$ through a semiring-isomorphism which sends the matrix $([a_{ij}])$ in $M_n(S/\bar{\Delta}_S)$ to the element $[(a_{ij})]$ in $M_n(S)/\bar{\Delta}_{M_n(S)}$. Also routine computations prove the following:

LEMMA 1.10. *Let S be a semiring. Let H be a left h -ideal of S . Then $H' = \{[x] \in S/\bar{\Delta}_S : x \in H\}$ is a left k -ideal of $S/\bar{\Delta}_S$. Conversely, if K is a left k -ideal of $S/\bar{\Delta}_S$, then $K_0 = \{x \in S : [x] \in K\}$ is a left h -ideal of S . Moreover, one has $(H')_0 = H$ and $(K_0)' = K$ and hence a bijective correspondence between the sets of all left h -ideals of S and all left k -ideals of $S/\bar{\Delta}_S$. In particular S is h -artinian iff $S/\bar{\Delta}_S$ is k -artinian.*

COROLLARY 1.11. *Let S be a semiring with multiplicative identity 1. Then the following three statements are equivalent ($n > 1$):*

- (i) $M_n(S)$ is an h -artinian semiring.
- (ii) S^n is an h -artinian S -semimodule.
- (iii) $\tilde{S}$ is an artinian ring.

PROOF. (i) $\Leftrightarrow$ (ii) follows from Corollary 1.3.

(i) $\Leftrightarrow$ (iii):

$M_n(S)$ is h -artinian

$\Leftrightarrow M_n(S)/\bar{\Delta}_{M_n(S)}$ is k -artinian (by Lemma 1.10)

$\Leftrightarrow M_n(S/\bar{\Delta}_S)$ is k -artinian (by the above isomorphism)

$\Leftrightarrow \tilde{S} = D(S/\bar{\Delta}_S)$ is artinian (by Theorem 1.9) ■

2. HILBERT BASIS THEOREM

DEFINITION 2.1 A semiring S is called *(left) noetherian* [*k-noetherian*, *h-noetherian*] if it satisfies the ascending chain condition on left ideals [*k-ideals*, *h-ideals*] of S .

It is clear that every *k-noetherian* semiring is *h-noetherian*. But the following example shows that the converse is not true.

EXAMPLE 2.2. Let Z_0^+ be the set of non-negative integers. Then $(Z_0^+, \max., \min.)$ is an *h-noetherian* semiring, but not *k-noetherian*.

Let S be a semiring and $A \subseteq S$. The smallest left *h-ideal* of S containing A is called the left *h-ideal* of S generated by A . The following lemma is obvious :

LEMMA 2.3. Let $A = \{ a_i \in S : i = 1, 2, \dots, n \}$ and

$$(A)_h = \left\{ x \in S : x = \sum_{i=1}^n c_i a_i + \sum_{i=1}^n n_i a_i + r = \sum_{i=1}^n \bar{c}_i a_i + \sum_{i=1}^n \bar{n}_i a_i + r, \text{ for some } c_i, \bar{c}_i, r \in S \text{ and } n_i, \bar{n}_i \in Z_0^+, \right. \\ \left. i = 1, 2, \dots, n \right\}$$

Then $(A)_h$ is the left *h-ideal* of S generated by A .

One can easily prove the following statements :

THEOREM 2.4. The following three conditions on left *h-ideals* of a semiring S are equivalent :

- (i) S is *h-noetherian*.
- (ii) Every non-empty set of left *h-ideals* of S has a maximal element.
- (iii) Every left *h-ideal* of S is finitely generated, i.e., for any left *h-ideal* I of S , there is a finite set $A \subseteq I$ such that $I = (A)_h$.

LEMMA 2.5. Any homomorphic image of an *h-noetherian* semiring is *h-noetherian*.

A halfring H is called unital [1] if $D(H)$ is a ring with identity. Stone [1] has obtained the following analogue of the Hilbert basis theorem for halfrings :

Let H be a unital halfring. Then $H[x]$ is *k-noetherian* iff $D(H)$ is *noetherian*.

We first show that the condition "unital" is not essential.

THEOREM 2.6. Let H be a halfring. Then $H[x]$ is *k-noetherian* iff $D(H)$ is *noetherian*.

PROOF. Let $D(H)$ be *noetherian*. Then $H[x]$ is *k-noetherian* [1].

Conversely, let $H[x]$ be *k-noetherian*. We define a mapping $\Psi : H[x] \longrightarrow D(H)$ by $\Psi(p(x)) = p_0 - p_1 + p_2 - p_3 + \dots$, for each $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \dots + p_n x^n$. Clearly, Ψ is a well-defined semiring-homomorphism. Also let $u \in D(H)$. Then $u = a - b$, $a, b \in H$. Now $\Psi(a + bx) = u$. Thus Ψ is surjective and hence $D(H)$ is *noetherian* ■

EXAMPLE 2.7. [1] Let S be the halfring described in the Example 1.4. Then S is *k-noetherian*. But $D(S)$ is not *noetherian*.

To prove the main result of this section we note that there is a semiring-isomorphism Ψ on $(S/\bar{\Delta}_S)[x]$ onto $(S[x])/\bar{\Delta}_{S[x]}$ defined by

$$\Psi([p_0] + [p_1]x + [p_2]x^2 + \dots + [p_n]x^n) = [p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n]$$

THEOREM 2.8. Let S be a semiring. Then $S[x]$ is *h-noetherian* iff $\bar{S}$ is *noetherian*.

PROOF. $S[x]$ is *h-noetherian*.

$\Leftrightarrow (S[x])/\bar{\Delta}S[x]$ is k -noetherian (by Lemma 1.10)

$\Leftrightarrow (S/\bar{\Delta}S[x])[x]$ is k -noetherian (by the above said isomorphism)

$\Leftrightarrow \tilde{S} = D(S/\bar{\Delta}_s)$ is noetherian (by Theorem 2.6) ■

ACKNOWLEDGEMENT. Thanks to the learned referee for all the pains undertaken for the improvement of the paper.

REFERENCES

1. STONE, H.E., Ideals in half-rings, Proc. of the Amer. Math. Soc., 33:1 (1972), 8-14.
2. LA TORRE, D.R., On h -ideals and k -ideals in hemirings, Publ. Math. Debrecen, 12 (1965), 219-226.
3. IIZUKA, K., On the Jacobson radical of a semiring, Tohoku Math. J., 11:2 (1959), 409-421.
4. WEINERT, H.J., Über Halbringe und Halbkörper II, Acta Math. Acad. Sci. Hungar. 14 (1963), 209-227.
5. STONE, H.E., Matrix representation of simple half-rings, Trans. Amer. Math. Soc., 233 (1977), 339-353.