THE $\Theta$-TRANSFORMATION OF
CERTAIN POSITIVE LINEAR OPERATORS

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ABSTRACT. The intention of this paper is to describe a construction method for a new sequence of linear positive operators, which enables us to get a pointwise order of approximation regarding the polynomial summator operators which have "best" properties of approximation.

KEY WORDS AND PHRASES. Approximation by positive linear operators, discrete linear operators, $(C, 1)$ means of Chebyshev series.

1. The aim of this paper can be described in the following way: Starting with a sequence $A = (A_n)$ of approximation operators, we construct -- by means of the so called $\Theta$ - transformation -- a new sequence of operators $B = (B_n) = \Theta (A)$.

With the known properties of $A$ we get the corresponding properties of the sequence $B = \Theta (A)$. We also prove, that if $A$ is the sequence of $(C, 1)$ - means of Chebyshev series, the polynomials $(B_n f)$, $f \in C(I)$, furnish a pointwise order of approximation similar to the best order of approximation.

Let $\Pi_n$, $n \in \mathbb{N}_0$, be the linear space of all algebraic polynomials with real coefficients of degree $\leq n$ and $T_n(t) = \cos (n \arccos t)$ the $n$ - th Chebyshev polynomial, $n \in \mathbb{N}_0$.

We denote by $X$ the normed linear spaces $C(I), I := [-1, 1]$ or $L_p^\omega (I), 1 \leq p < \infty$, endowed with norms $\|f\|_{C(I)} = \|f\| : = \max_{t \in I} |f(t)|$ for $f \in C(I)$, respectively $\|f\|_p = \left[ \frac{1}{-1} \int_{-1}^{1} |f(t)|^p \omega(t) \, dt \right]^{1/p}$, where $f$ is an element of the Lebesgue space $L_p^\omega (I)$ with the weight $\omega(t) = \frac{1}{\sqrt{1-t^2}}$.

Further for $f \in X$ and a polynomial $g$ we use the inner product

$$<f, g> = \frac{1}{-1} \int_{-1}^{1} f(t) g(t) \omega(t) \, dt.$$

The translation operator $\tau_x : X \rightarrow X, x \in I$, defined by

$$(\tau_x f)(t) = \frac{1}{2} \left[ f(x t + \sqrt{1-x^2} \sqrt{1-t^2}) + f(x t - \sqrt{1-x^2} \sqrt{1-t^2}) \right], \quad (t, x) \in I \times I,$$

has the property

$$(\tau_x T_k)(t) = T_k(t) T_k(x), \quad k \in \mathbb{N}$$

(see [3]).

If we use the convolution product $\star : L_1^\omega (I) \times L_1^\omega (I) \rightarrow L_1^\omega (I)$

$$f \ast g)(x) = \frac{1}{-1} \int_{-1}^{1} f(t) (\tau_x g)(t) \omega(t) \, dt,$$
then our aim is to construct some approximation operators \( A_n : X \to \Pi_n, n \in \mathbb{N} \), such that
\[
\lim_{n \to \infty} \| f - A_n f \|_X = 0, \quad f \in X.
\]

A sequence \( a = (a_n)_{n \in \mathbb{N}_0}, a_n \in \Pi_n \), with degree \( a_n = n \) for all \( n \in \mathbb{N}_0 \), is called a polynomial sequence. If \( \mathcal{P}^+ \) denotes the set of all polynomial sequences \( a = (a_n)_{n \in \mathbb{N}_0} \) with the properties
\[
\begin{align*}
1. & \quad a_n(x) \geq 0, \quad x \in I \\
2. & \quad (1, a_n) = 1, \quad n \in \mathbb{N}_0.
\end{align*}
\]
then for
\[
a_n(x) = \sum_{k=0}^{n} \omega_k \alpha_{k,n} T_k(x), \quad \text{where} \quad \omega_0 = \frac{1}{\pi}, \quad \omega_k = \frac{2}{\pi}, \quad k \geq 1, \quad (1.1)
\]
\[
a_n(x, t) = (\tau_n a_n)(t) = \sum_{k=0}^{n} \omega_k \alpha_{k,n} T_k(x) T_k(t).
\]
we consider the sequence \( A := A(a) = (A_n)_{n \in \mathbb{N}_0}, A_n : X \to \Pi_n \), of linear positive operators, defined by \( A_n f = f * a_n = a_n * f \) that is
\[
(A_n f)(x) = A_n(f; x) = \sum_{k=0}^{n} \omega_k \alpha_{k,n} (f, T_k) T_k(x) = \int_{-1}^{1} a_n(x, t) f(t) \omega(t) dt, \quad x \in I. \quad (1.2)
\]
In this case \( a = (a_n)_{n \in \mathbb{N}_0} \) is called the generating sequence of \( A = (A_n) \).

If \( A(a) = (A_n) \) is defined as in (1.1) and (1.2), then \( |\alpha_{k,n}| = \|(T_k, a_n)\| \leq 1 \) and let us define the functionals \( r_n : \mathcal{P}^+ \to \mathbb{R}, n \in \mathbb{N}, \)
\[
r_n(A) := 1 - \alpha_{1,n} = 1 - (T_1, a_n), \quad n \in \mathbb{N}.
\]
An important polynomial sequence \( \varphi = (\varphi_n)_{n \in \mathbb{N}_0}, \varphi \in \mathcal{P}^+ \), was considered by L.Fejer, namely
\[
\varphi_n(x) = \frac{1 - T_{n+1}}{\pi(n+1)(1-x)} = \sum_{k=0}^{n} \omega_k \left(1 - \frac{k}{n+1}\right) T_k(x). \quad (1.3)
\]
The corresponding linear positive operators \( F = (F_n)_{n \in \mathbb{N}_0}, F_n = f * \varphi_n \) are the \((C,1)\) – means of Chebyshev series, i.e. the Fejer operators \( F_n : X \to \Pi_n, n \in \mathbb{N}_0, \)
\[
(F_n f)(x) = \sum_{k=0}^{n} \omega_k \left(1 - \frac{k}{n+1}\right) (f, T_k) T_k(x), \quad f \in X. \quad (1.4)
\]
There exists a connection between the operators defined in (1.2) and those from (1.4). Indeed, using the equalities \( A_n T_k = \alpha_{k,n} T_k, k \in \mathbb{N}_0 \), we get with
\[
a_n = \sum_{k=0}^{n} \omega_k \alpha_{k,n} T_k = (n + 1) A_n \varphi_n - n A_n \varphi_{n-1}
\]
the identity
\[
A_n f = (n + 1)a_n * F_n f - na_n * F_{n-1} f.
\]

2. Let \( b = (b_n)_{n \in \mathbb{N}_0} \) be an element from \( \mathcal{P}^+ \) with
\[
b_n(x) = \sum_{k=0}^{n} \omega_k \beta_{k,n} T_k(x), \quad (2.1)
\]
and \( B = B(b) = (B_n)_{n \in \mathbb{N}}, B_n : X \to \Pi_n, \) the operators with the "generating polynomial sequence \( b"", defined by
\[
(B_n f)(x) = \sum_{k=0}^{n} \omega_k \beta_{k,n} (f, T_k) T_k(x), \quad x \in I. \quad (2.2)
\]
Suppose that
\[
\int_{-1}^{1} h(t) \omega(t) dt = \sum_{k=1}^{m(n)} c_k(n) h(z_k), \quad (2.3)
\]
with \( c_k(n) \geq 0, z_k \in [-1,1], k = 1,2, \ldots, m(n) \), is a quadrature formula which is exact for all polynomials \( h \in \Pi_{s(n)} \) with \( s(n) \geq n + 2, n \in \mathbb{N} \).

For \( b = (b_n) \in \mathcal{P}^+ \) and \( B = (B_n) \) as in (2.1) - (2.2) we consider the linear positive operators \( B_n^*, \hat{B}_n, n \in \mathbb{N}_0 \), where for \( f \in X \)
\[
(\hat{B}_n f)(x) = \sum_{k=1}^{m(n)} c_k(n) (\tau_k b_n)(z_k) f(z_k) \quad (2.4)
\]
and
\[
(\hat{B}_n f)(x) = \sum_{k=1}^{m(n)} c_k(n) (\tau_k f b_n)(z_k). \quad (2.5)
\]
The sequence \( B^* = (B_n^*) \) is called "the discrete form" of \( B = (B_n) \), with respect to (2.3). The operator \( \hat{B}_n \) appears to be useful for the connection between \( B_n \) and \( B_n^* \).

**Lemma 2.1** If \( \hat{B}_n \) is defined as in (2.5), then for \( j \in \{1,2\} \)
\[
\hat{B}_n(1 - t^j; x) = \frac{1}{2^{j-1}} (1 - \beta_{j,n}). \quad (2.6)
\]

**Proof:** Let us observe that
\[
\int_{-1}^{1} (1 - t^j)b_n(t) T_k(t) \omega(t) dt = B_n((1 - t^j) T_k(t); 1).
\]
Therefore
\[
\tau_k((1 - t^j)b_n(t))(z) = \sum_{k=0}^{n+j} \omega_k B_n((1 - t^j) T_k(t); 1) T_k(x) T_k(z)
\]
and using (2.3) for \( j \in \{1,2\} \) we have
\[
\hat{B}_n(1 - t^j; x) = \sum_{k=1}^{m(n)} c_k(n) \tau_k((1 - t^j)b_n(t))(z_k)
\]
\[
= \int_{-1}^{1} \tau_k((1 - t^j)b_n(t))(x) \omega(z) dz = B_n(1 - t^j; 1).
\]
Finally
\[
B_n(1 - t; 1) = 1 - \beta_{1,n}, \quad B_n(1 - t^2; 1) = \frac{1}{2} (1 - \beta_{2,n}), \quad (2.7)
\]
which completes the proof.

**Theorem 2.2** Suppose that \( B_n \) is defined by means of (2.2) with \( b \in \mathcal{P}^+ \). Let \( B_n^* \) be the discrete operator from (2.4) and
\[
\delta_n(x) \quad \text{one of the functions} \quad B_n(|x - t||; x) \quad \text{or} \quad B_n^*(|x - t||; x).
\]

Then for \( x \in I \)
\[
|x| r_n(B) \leq \delta_n(x) \leq \sqrt{1 - x^2} \sqrt{1 - \beta_{2,n}} + |x| r_n(B). \quad (2.8)
\]
Proof: With \( e_k(t) = t^k, k \in \mathbb{N}_0 \), it is known that for convex functions \( \gamma \in C(I) \) we have

\[
\gamma(Le_k) \leq L \gamma \quad \text{on} \quad I, \tag{2.9}
\]

where \( L \) is a linear positive operator \( C(I) \to C(I) \) with \( Le_0 = e_0 \) (see [8]). If we select \( \gamma(t) = |x - t| \), \( L = B_n \), we have by using the inequality (2.9)

\[
|x - x \beta_{1,n}| \leq B_n(|x - t|; x);
\]
or on the other hand for \( L = B_n^* \)

\[
|x - (B_n^* e_1)(x)| \leq B_n^* (|x - t|; x).
\]

For \( h \in \Pi_2 \) it is \( B_n^* h = B_n h \) and so we obtain the lower bound in (2.8).

Further let us denote

\[
\psi_1(x, t) = xt + \sqrt{1 - x^2} \sqrt{1 - t^2}
\]

\[
\psi_2(x, t) = xt - \sqrt{1 - x^2} \sqrt{1 - t^2}.
\]

Then for \( x, t \in I, j \in \{1, 2\} \)

\[
|x - \psi_j(x, t)| \leq \sqrt{1 - x^2} \sqrt{1 - t^2} + |x|(1 - t) \tag{2.10}
\]

and

\[
|x - t| \leq \sqrt{1 - x^2} \sqrt{1 - \psi_j^2(x, t)} + |x|(1 - \psi_j(x, t)). \tag{2.11}
\]

Define the linear positive functionals \( J_n : C(I) \to \mathbb{R}, n \in \mathbb{N}_0, \) by \( J_n(f) = \langle f, b_n \rangle \).

We have

\[
J_n(1 - t') = B_n(1 - t'; 1)
\]

more precisely (see (2.7))

\[
J_n(1 - t) = 1 - \beta_{1,n}
\]

\[
J_n(\sqrt{1 - t^2}) \leq \sqrt{J_n(1 - t^2)} = \sqrt{\frac{1 - \beta_{2,n}}{2}}.
\]

Because

\[
(B_n f)(x) = \int_{-1}^{1} b_n(t)(\tau x f)(t) \omega(t) dt
\]

and (2.10) enables us to write

\[
\tau_x(|x - .; t) = \left| x - \frac{\psi_1(x, t) + \psi_2(x, t)}{2} \right| \leq \sqrt{1 - x^2} \sqrt{1 - t^2} + |x|(1 - t)
\]

one finds

\[
B_n(|x - t|; x) \leq \sqrt{1 - x^2} J_n(\sqrt{1 - t^2}) + |x| J_n(1 - t)
\]

\[
\leq \sqrt{1 - x^2} \sqrt{\frac{1 - \beta_{2,n}}{2}} + |x|(1 - \beta_{1,n}),
\]

i.e. the upper bound in (2.8). Regarding the discrete operators \( (B_n^*) \), we have from (2.4) and (2.11)
\[ B_n^*([x-t]; x) = \sum_{k=1}^{m(n)} c_k(n) |x - z_k| b_n(\psi_1(x, z_k)) + b_n(\psi_2(x, z_k)) \]

\[ \leq \sum_{k=1}^{m(n)} c_k(n) \frac{\sqrt{1-x^2} \sqrt{1-\psi_1^2(x, z_k)}}{2} + |x|(1 - \psi_1(x, z_k)) b_n(\psi_1(x, z_k)) \]

\[ + \sum_{k=1}^{m(n)} c_k(n) \frac{\sqrt{1-x^2} \sqrt{1-\psi_2^2(x, z_k)}}{2} + |x|(1 - \psi_2(x, z_k)) b_n(\psi_2(x, z_k)) \]

\[ = \sqrt{1-x^2} \sum_{k=1}^{m(n)} c_k(n) r_2(\sqrt{1-t^2} b_n(t))(z_k) \]

\[ + \sum_{k=1}^{m(n)} c_k(n) r_2((1-t)b_n(t))(z_k) \]

\[ = \sqrt{1-x^2} B_n(\sqrt{1-t^2}; x) + |x| B_n(1-t; x). \]

From (2.6) using Schwarz inequality we complete the proof. □

Other upper bounds for \( \delta_n \) were obtained by J.D. Cao and H.H. Gonska [5].

**Theorem 2.3** Let \( b = (b_n) \) be an arbitrary polynomial sequence from \( \mathcal{P}^+ \). Suppose that \( B = (B_n), B^* = (B_n^*) \) are defined as in (2.2) respectively (2.4). Then for \( f \in C(I), x \in I \),

\[ |f(x) - (B_n f)(x)| \leq 2 \omega(f; \nabla^B_n(x)) \leq 4 \omega(f; \Delta^B_n(x)) \] \quad (2.12)

\[ |f(x) - (B_n^* f)(x)| \leq 2 \omega(f; \nabla^{B^*}_n(x)) \leq 4 \omega(f; \Delta^{B^*}_n(x)) \] \quad (2.13)

where \( \omega(f; \delta) := \sup \{|f(t+h) - f(t)|; |h| \leq \delta, t, t+h \in I \} \) and

\[ \nabla^B_n(x) = \sqrt{1-x^2} \sqrt{1-\beta_{1,n}} + |x|(1 - \beta_{1,n}) \]

\[ \Delta^B_n(x) = \sqrt{(1-x^2)(1-\beta_{1,n})} + |x|(1 - \beta_{1,n}), \]

with \( \beta_{1,n} = (B_n e_1)(1) \).

**Proof:** It is known that an arbitrary linear positive operator \( L_n : C(I) \to C(I) \) with \( L_n e_0 = e_0 \) satisfies the inequality

\[ |f(x) - (L_n f)(x)| \leq 2 \omega(f; L_n([x-t]; x)). \]

The upper estimate from theorem 2.2 enables us to write

\[ |f(x) - (L_n f)(x)| \leq 2 \omega(f; \nabla^B_n(x)), \quad x \in I, \]

where \( L_n \) is one of the operators \( B_n \) or \( B_n^* \).
Let \( q_m(t) = (1 - t)^m \), \( m \in \mathbb{N} \), and observe that \( q_j, q_m \) are monotone on \( I \) in the same sense. By means of Chebyshev inequality we have \( (B_n q_j) (x) (B_n q_m) (x) \leq (B_n q_{j+m}) (x) \), \( j, m \in \mathbb{N}_0 \), where we find with \( j = m = 1 \) and \( x = 1 \)

\[
0 \leq 1 - \beta_{j,m} \leq 2(1 + \beta_{j,m})r_n(B) \leq 4r_n(B). \tag{2.14}
\]

Therefore
\[
\omega(f; \nabla_n^B(x)) \leq \omega(f; \sqrt{2} \Delta_n^B(x)) \leq 2 \omega(f; \Delta_n^B(x)), \quad x \in I.
\]

which proves this theorem. \( \Box \)

**Remark:** One knows that for \((b_n) \in \mathcal{P}^+ \) the Fejér inequality \([6]\) holds

\[
\beta_{1,n} \leq \cos \frac{\pi}{n + 2}, \quad n \in \mathbb{N}.
\]

In the case of Jacobi polynomials \( R^{(\alpha, \beta)}_n \), \( \alpha \geq \beta \geq -\frac{1}{2} \), for an arbitrary \( n \) a similar extremal problem is solved in \([8]\). For an even \( n \) the problem is considered in \([1]\), p.68.

However, for all linear positive operators \( B = (B_n) \) generated by polynomial sequences \( b = (b_n) \in \mathcal{P}^+ \) one has

\[
r_n(B) \geq 2\sin\frac{\pi}{2(n + 2)}. \tag{2.15}
\]

Let us present a short proof of Fejér’s inequality (2.15). If \( h \in \Pi_{n+1} \), then it is easy to observe that

\[
\langle 1, h \rangle = \sum_{k=0}^{n} c_k h(x_{k,n}), \quad s = \left\lfloor \frac{n}{2} \right\rfloor + 1,
\]

\( x_{0,n} = -1, \quad x_{k,n} = \cos \frac{2k-1}{n+2} \pi, \quad k \geq 1, \quad c_0 = 2x, \quad c_1 = \frac{1 - (-1)^n}{4}, \quad c_i = \cdots = c_s = \frac{2x}{n+2} \).

If \( h_0(t) = (1 - t)b_n(t) \) then

\[
r_n(B) = \langle 1, h_0 \rangle = \sum_{k=0}^{n} c_k (1 - x_{k,n})b_n(x_{k,n}) \geq c_1 (1 - x_{1,n})b_n(x_{1,n}) \geq 1 - x_{1,n} = 2\sin^2 \frac{\pi}{2(n + 2)}.
\]

Therefore the equality holds if and only if

\[
b_n(x) = b_n^*(x) = \lambda_n (x + 1)^d \prod_{k=2}^{s} (x - x_{k,n})^2, \quad d = \left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor,
\]

where \( \lambda_n \) is selected such that \( b_n(x_{1,n}) = \frac{n+2}{4x} \). It may be shown that \([9]\)

\[
b_n^*(x) = \kappa_n \frac{1 + T_{n+2}(x)}{(x - \cos \frac{\pi}{n+2})^2}, \quad \kappa_n = \frac{1}{\pi(n+2)} \sin^2 \frac{\pi}{n + 2}.
\]

3. A polynomial sequence \( a = (a_n) \) belongs to the class \( \mathcal{P}^1 \) if and only if

i) \( a \in \mathcal{P}^+ \) and

ii) for each \( n \in \mathbb{N} \) there exists at least a root \( z_0(n) \) of \( a_n \) in \( I \).

We denote \( z_0 = z_0(n+1) \) and remind that \( a_n(x, t) = (\tau_x a_n)(t), \quad a_{n+1}(z_0) = a_{n+1}(1, z_0) = 0 \). Define \( b = (b_n) \) to be the sequence of polynomials

\[
b_n(x) = \frac{1}{c_n} a_{n+1}(x, z_0), \quad (3.1)
\]
where
\[ c_n = \int_{-1}^{1} \frac{a_{n+1}(t, z_0)}{1 - t} \omega(t) dt. \]

It is clear that the positivity of the translation operator certifies the fact that \( b = (b_n) \in \mathcal{P}^+ \). If \( l : \mathcal{P}^1 \to \mathcal{P}^+ \) is the mapping \( (a_n) \to (b_n) \), then we write \( b = l(a) \).

**Definition** If \( a = (a_n) \in \mathcal{P}^1, b = (b_n) = l(a) \), then the sequence \( B = (B_n) \) defined in (2.2) is called the \( \Theta \)-transformation of the sequence \( A = (A_n) \) from (1.2) and we write \( B = \Theta(A) \).

**Lemma 3.1** Suppose that \( a = (a_n) \in \mathcal{P}^1 \),
\[ a_n(x) = \sum_{k=0}^{n} \omega_k \alpha_{k,n} T_k(x), \quad a_{n+1}(z_0) = 0, \quad z_0 = z_0(n+1) \in I, \]
is the generating polynomial sequence for the operators \( A = (A_n) \).

If \( b = (b_n) = l(a) \), then
\[ b_n(x) = -\frac{2}{c_n} \sum_{k=0}^{n} (k+1) \alpha_{k+1,n+1} T_{k+1}(z_0) \varphi_k(x), \quad n \in \mathbb{N}, \]
where \( \varphi_k \) is defined in (1.3).

**Proof:** Let \( d_k(t,x) \) be the Dirichlet kernel
\[ d_k(t,x) = \sum_{j=0}^{k} \omega_j T_j(t) T_j(x) \]
and \( S_n : X \to \Pi_n \) be the partial sum of Chebyshev series, i.e.
\[ (S_n f)(x) = \sum_{j=0}^{n} \omega_j f(T_j) T_j(x) = \int_{-1}^{1} d_n(t,x) f(t) \omega(t) dt. \quad (3.2) \]

From (3.1) we get
\[ b_n(x) = \frac{1}{c_n} \frac{a_{n+1}(x, z_0) - a_{n+1}(1, z_0)}{1 - x} = -\frac{1}{c_n} \sum_{k=1}^{n+1} \omega_k \alpha_{k,n+1} \frac{1 - T_k(x)}{1 - x} T_k(z_0) \]
\[ = -\frac{2}{c_n} \sum_{k=0}^{n} (k+1) \alpha_{k+1,n+1} T_{k+1}(z_0) \varphi_k(x). \]

Further, we may write
\[ b_n(x) = -\frac{2}{c_n} \sum_{k=0}^{n} (k+1) \alpha_{k+1,n+1} T_{k+1}(z_0) \sum_{j=0}^{k} \omega_j (1 - \frac{j}{k+1}) T_j(x) = \sum_{k=0}^{n} \omega_k \beta_{k,n} T_k(x) \]
with
\[ \beta_{k,n} = -\frac{2}{c_n} \sum_{j=k}^{n} (j+1-k) \alpha_{j+1,n+1} T_{j+1}(z_0) \quad (3.3) \]
Now, if \( \varphi_k(t, x) = (\tau_x \varphi_k)(t) \)

\[
\sum_{j=k}^{n} (j + 1 - k)T_{j+1}(t)T_{j+1}(z_0) = \frac{n}{2} (n + 2 - k)d_{n+1}(t, z_0) - d_k(t, z_0) + (k + 1)\varphi_k(t, z_0) - (n + 2)\varphi_{n+1}(t, z_0).
\]

Using (1.4), (3.2) - (3.3) we conclude with

**Lemma 3.2** Under the hypothesis of lemma 3.1 the coefficients \( \beta_{k,n} \) in

\[
b_n = \sum_{k=0}^{n} \omega_k \beta_{k,n} T_k
\]

are

\[
\beta_{k,n} = \frac{\pi}{c_n} \left( (n + 2)(F_{n+1} a_{n+1})(z_0) - (k + 1)(F_{n+1} a_{n+1})(z_0) + (S_{n+1})(z_0) \right), \quad \text{(3.4)}
\]

where

\[
c_n = \pi(n + 2)(F_{n+1} a_{n+1})(z_0). \quad \text{(3.5)}
\]

By considering the family of linear operators \( I_{k,n}, k = 0, 1, \ldots, n, n \in \mathbb{N}, \) defined on \( P^1 \) by

\[
I_{k,n} := (n + 2)F_{n+1} - (k + 1)F_k + S_k
\]

one finds the operational formula

\[
\beta_{k,n} = \frac{(I_{k,n} a_{n+1})(z_0)}{(n + 2)(F_{n+1} a_{n+1})(z_0)}, \quad k = 0, 1, \ldots, n. \quad \text{(3.6)}
\]

Let us note that if \( B = \Theta(A) \), then

\[
r_n(B) = 1 - \beta_{1,n} = \frac{1}{\pi(n + 2)(F_{n+1} a_{n+1})(z_0)}.
\]

Using the above results one can formulate the following

**Theorem 3.3** Let \( a = (a_n) \in P^1, b = (b_n) = l(a) \in P^+ \) and \( B = \Theta(A) \). If

\[
m_{k,n} = -\frac{2}{c_n} (k + 1)a_{k+1,n}T_{k+1}(z_0), \quad c_n = \pi(n + 2)(F_{n+1} a_{n+1})(z_0)
\]

then \( B = (B_n) \) is a summability method of Fejér operators \( F = (F_n) \), more precisely

\[
B_n = \sum_{k=0}^{n} m_{k,n} F_k.
\]

Moreover, for all \( x \in I \) and \( f \in C(I) \)

\[
|f(x) - (B_n f)(x)| \leq 4 \omega \left( f; \frac{|x|}{c_n} + \sqrt{\frac{1 - x^2}{c_n}} \right), \quad n \in \mathbb{N}.
\]

4. In this section we will consider the case \( a = \varphi = (\varphi_n) \), with \( \varphi_n \) being as in (1.3) and \( z_0 = z_0(n + 1) = \cos \frac{2\pi}{n+2} \). At first we observe in our case

\[
c_n = \pi(n + 2)(F_{n+1} \varphi_{n+1})(z_0) = \pi(n + 2) \sum_{k=0}^{n} \omega_k \left( 1 - \frac{k}{n + 2} \right)^2 \cos \frac{2k\pi}{n + 2}
\]
that is
\[ \frac{1}{c_n} := r_n(B) = \sin^2 \pi \frac{n}{n + 2}. \] (4.1)

If we select in (3.3) \( a_{k+1, n+1} = 1 - \frac{k+1}{n+2} \), \( z_0 = \cos \frac{2\pi}{n+2} \) or in (3.6) \( a_{n+1} = \varphi_{n+1} \), one finds the following

**Lemma 4.1** If \( b = (b_n) = l(\varphi), \varphi = (\varphi_n) \), then
\[ b_n(x) = \sum_{k=0}^{n} \omega_k \beta_{k,n} T_k(x) \]

with
\[ \beta_{k,n} = \frac{n - k + 2}{n + 2} \cos^2 \frac{k\pi}{n + 2} + \frac{\cos \frac{2\pi}{n+2}}{(n+2) \sin \frac{\pi}{n+2}} \cos \frac{k\pi}{n+2} \sin \frac{k\pi}{n+2}. \] (4.2)

Moreover
\[ b_n(x) = \frac{\kappa_n}{(1 - x \cos \frac{2\pi}{n+2}) \left(1 - T_{n+2}(x)\right)} \left(1 - x \cos \frac{2\pi}{n+2}\right)^2 \] (4.3)

where
\[ \kappa_n = \frac{1}{\sin^2 \pi \frac{n}{n + 2}} \] (4.4)

Further, let

\[ B = (B_n) = \Theta(F), \]

where \( F = (F_n) \) is the sequence of Fejér operators.

If \( f \in X \) and \( \beta_{k,n} \), \( b_n \) are as in (4.2) - (4.4), then
\[ B_n f = \sum_{k=0}^{n} \omega_k \beta_{k,n} (f, T_k) T_k = f \ast b_n = b_n \ast f \] (4.5)

and also, if \( \tilde{m}_{k,n} = 2\pi \kappa_n (k+1)(k-n-1) \cos \frac{2(k+1)x}{n+2} \) then
\[ B_n f = \sum_{k=0}^{n} \tilde{m}_{k,n} F_k f. \] (4.6)

We note that the coefficients \( \tilde{m}_{k,n} \) satisfy \( \tilde{m}_{k,n} = \tilde{m}_{n-k,n} \), \( k = 0, 1, \ldots, n \).

In order to obtain a discrete form of the operators \( B = (B_n) \) defined by (4.5) let us observe that the translation of \( b_n \) from (4.3) is
\[ (\tau_x b_n)(y) = \kappa_n \frac{v_n(x; y) \left(1 - T_{n+2}(x)T_{n+2}(y)\right) - w_n(x; y)(1 - x^2)(1 - y^2)U_{n+1}(x)U_{n+1}(y)}{(x - y)^2 \left((x - y \cos \frac{2\pi}{n+2})^2 - (1 - y^2)\sin^2 \frac{2\pi}{n+2}\right)^2} \] where
\[ v_n(x; y) = (1 - xy)(\tau_x p)(y) + (1 - x^2)(1 - y^2) \cos \frac{2\pi}{n+2} \left[(x - y)^2 - (2xy - 1 - \cos \frac{2\pi}{n+2})\right]^2 \] (4.7)

\[ w_n(x; y) = (x - y)^2 - (2xy - 1 - \cos \frac{2\pi}{n+2})^2 + \cos^2 \frac{2\pi}{n+2} (\tau_x p)(y) \]

with \( U_{n+1}(x) = \frac{\sin(n+2)\arccos x}{\sqrt{1-x^2}} \) and \( p(x) = (1 - x)(x - \cos \frac{2\pi}{n+2})^2 \).
If in quadrature formula (2.3) we choose the knots \( z_k = z_{k,n} \) such that \( U_{n+1}(z_k) = 0 \), then the polynomials \((\tau_x b_n) (z_{k,n})\) have a simpler form. Therefore, we will consider the Bouzitat formula of the second kind

\[
\int_{-1}^{1} g(t) \omega(t) dt = \sum_{k=0}^{n+2} c_k(n) g(z_{k,n}) - \frac{\pi}{2^{2n+3}(2n+4)!} g^{(2n+4)}(\xi_n), \quad g \in C^{(2n+4)}(I), \quad \xi_n \in I,
\]

with \(c_0(n) = c_{n+2}(n) = \frac{\pi}{2(n+2)}, \quad c_1(n) = \cdots = c_{n+1}(n) = \frac{\pi}{n+2}, \quad z_{k,n} = \cos \frac{k\pi}{n+2}, \quad k \in \mathbb{Z}.

In conclusion let \( B_n : X \to \Pi_n, \quad n \in \mathbb{N}_0 \), be the linear positive operators with the images

\[
(B_n f) (x) = \frac{\pi}{n+2} \left( \frac{f(-1)b_n(-x) + f(1)b_n(x)}{2} + \sum_{k=1}^{n+1} v_n(x; z_{k,n}) \frac{1 - (-1)^k T_{n+2}(x)}{(x - z_{k,n})^2(x - z_{k-2,n})^2(x - z_{k+2,n})^2} f(z_{k,n}) \right); \quad (4.8)
\]

the polynomials \( v_n \) being explained in (4.7).

Another representation of the operator \( B_n \) may be obtained in the following way. Let us consider the bilinear form for \( f, g : I \to \mathbb{R} \)

\[
(f,g)_{n+1} = \frac{\pi}{n+2} \left( \frac{f(-1)g(-1) + f(1)g(1)}{2} + \sum_{k=1}^{n+1} f(\cos \frac{k\pi}{n+2}) g(\cos \frac{k\pi}{n+2}) \right).
\]

It is easy to see that \((f,g) = (f,g)_n\) for \( fg \in \Pi_{2n+3} \).

Now

\[
(B_n f) (x) = \sum_{k=0}^{n+2} c_k(n) f(z_{k,n}) (\tau_x b_n) (z_{k,n}) = \sum_{k=0}^{n+2} c_k(n) f(z_{k,n}) \sum_{j=0}^{n} \omega_j \beta_{j,n} T_j(x) T_j(z_{k,n})
\]

implies

\[
(B_n f) (x) = \sum_{j=0}^{n} \omega_j \beta_{j,n} (f, T_j)_n T_j(x),
\]

which is the discrete version of (4.5). Similar discrete approximation operators were studied by A.K. Varma and T.M. Mills [11]. They obtained such operators as a summability method of Lagrange interpolation.

By using (4.1) in (2.12) – (2.13) we obtain

**Theorem 4.2** Suppose that \( B = (B_n) \) is the \( \Theta \)–transformation of the Fejér operators \( F = (F_n) \). Let \( B^* = (B^*_n) \) be defined as in (4.8). If \( f \in C(I), \ x \in I, \) and

\[
\epsilon_n(x) = \sqrt{1 - x^2} \sin \frac{\pi}{n+2} + |x| \sin^2 \frac{\pi}{n+2}
\]

then for \( n \in \mathbb{N} \)

\[
|f(x) - (B_n f)(x)| \leq 4 \omega(f; \epsilon_n(x))
\]

\[
|f(x) - (B^*_n f)(x)| \leq 4 \omega(f; \epsilon_n(x)).
\]

**Remarks:**

- If \( B = (B_n) \) is the \( \Theta \)–transformation of \( F = (F_n) \), then
which means that the linear combination of Fejér operators \((4.6)\) approximates the functions from \(C(I)\) better than \(F_n f\).

- Note that the inequality
  \[
  \epsilon_n(x) < \pi^2 \left( \frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2} \right), \quad x \in I,
  \]
  furnishes an estimation of Timan's type
  \[
  |f(x) - (B_n^* f)(x)| \leq \epsilon_0 \omega(f; \frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2})
  \]
  \[x \in I, \ f \in C(I), \ n \in \mathbb{N}, \ \epsilon_0 \in (0, 40].\]

By means of the second order modulus of smoothness

\[
\omega_2(f, h) := \sup \{|f(x-\delta) - 2f(x) + f(x+\delta)|; x, x \pm \delta \in I, \ 0 \leq \delta \leq h\}, \ f \in C(I),
\]

one finds

**Theorem 4.3** Let \(B = (B_n) = \Theta(F)\) and \(B^* = (B_n^*)\) as in \((4.8)\). For \(f \in C(I), \ x \in I,\) we have

\[
|f(x) - (B_n f)(x)| \leq \epsilon_0 \left( \omega_2(f; \frac{1}{n}) + \frac{|x|}{n} \omega(f; \frac{1}{n}) \right),
\]

\[
|f(x) - (B_n^* f)(x)| \leq \epsilon_0 \left( \omega_2(f; \frac{1}{n}) + \frac{|x|}{n} \omega(f; \frac{1}{n}) \right),
\]

where \(\epsilon_0 = 3 + 2\pi^2\) and \(n \in \mathbb{N}\).

**Proof:** Let \(\Omega_{2,\alpha}(t) = (t-x)\beta^2\) then we get with \((4.1)\)

\[
(B_n \Omega_{2,x})(x) = (B_n^* \Omega_{2,x})(x)
\]

\[
= r_n(B) \left( 1 + \frac{n+1}{n+2} (1 - 2x^2) \cos \frac{2\pi}{n+2} \right)
\]

\[
< 2r_n(B) < \frac{2\pi^2}{n^2}.
\]

If \(L\) is a linear positive operator which preserves the constant functions, there is – according to H.H.Gonska ([7] theorem 2.4) – for \(h \in (0, 2]\) and \(x \in I,\)

\[
|f(x) - (L f)(x)| \leq \left( 3 + \frac{1}{h^2} (L \Omega_{2,x})(x) \right) \omega_2(f; h) + \frac{2}{h} |c_1(x) - (L c_1)(x)| \omega(f; h).
\]

Therefore, with \(h = \frac{1}{n}\) in our case we find the desired inequalities. □

Finally let us suppose that \(\delta \in (0, 1]\) and \(f \in Lip_2(\alpha, C), \ 0 < \alpha \leq 2.\) Then \(\omega_2(f; \delta) \leq C \delta^\alpha, \)

\[
C := const. \text{ and } \omega(f; \delta) \leq \begin{cases} 2\|f\|, & \alpha \in (0, 1] \smallskip \\ \delta\|f\|, & \alpha \in (1, 2] \end{cases}
\]
We get

\[ \delta \omega(f; \delta) \leq \begin{cases} 2 \| f \| \delta^\alpha, & \alpha \in (0, 1] \\ \| f' \| \delta^\alpha, & \alpha \in (1, 2] \end{cases}, \]

and so one finds a positive constant \( \mathcal{M} = M(f) \) such that

\[ \omega_2(f; \delta) + |x| \delta \omega(f; \delta) \leq M \delta^\alpha, \quad \alpha \in (0, 2], \delta \in (0, 1], \, x \in I. \]

If we choose \( \delta = \frac{1}{n} \), from Theorem 4.3 we get

\[ | f(x) - (B_n^* f)(x) | \leq \frac{M}{n^\alpha}, \quad x \in I. \]

In conclusion the linear summator operators \((B_n^*)\) have the co-domain in \( \Pi_n \) and satisfy

\[ \| f - B_n^* f \|_{C(I)} = \mathcal{O}(n^{-\alpha}) \]

provided \( f \in Lip_2(\alpha, C) \), \( 0 < \alpha \leq 2 \), i.e. an an answer to a problem proposed by P.L.Butzer [2]. Other solutions for Butzer’s problem are presented in [5]. However, some summability methods for Lagrange interpolation (see [11], [10]) furnish us also an affirmative answer to the question raised in [2].

References

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