COMPUTATION OF HILBERT SEQUENCE FOR COMPOSITE QUADRATIC EXTENSIONS USING DIFFERENT TYPE OF PRIMES IN \( \mathbb{Q} \)

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ABSTRACT. First, we will give all necessary definitions and theorems. Then the definition of a Hilbert sequence by using a Galois group is introduced. Then by using the Hilbert sequence, we will build tower fields for extension \( K/k \), where \( K = k(\sqrt{d_1}, \sqrt{d_2}) \) and \( k = \mathbb{Q} \) for different primes in \( \mathbb{Q} \).

KEY WORDS AND PHRASES: Composite quadratic extension, Hilbert sequence


1. INTRODUCTION

Let \( K/k \) be an extension of degree \( n \). We consider the tower of fields and a tower of integer rings for this extension
\[
\begin{align*}
K & \supseteq \ldots \supseteq L \ldots \supseteq k \\
O_K & \supseteq \ldots \supseteq O_L \ldots \supseteq O_k
\end{align*}
\] (1.1)

A prime ideal \( P \) in \( K \) determines a prime \( P_L \) in each field of the tower, where each \( P_L \) is divisible by \( P \). Let \( p \) be a rational prime that is divisible by all these prime ideals \( P_L \). Then we have:
\[
P_L = P_k \cap O_L, \quad p = P_L \cup \mathbb{Z}.
\]

If the prime ideal \( p \) in \( k \) does not split into \( n \) distinct factors of \( P \) in \( K \), how far can we go in terms of an intermediate field where splitting occurs? This will be answered later.

First we define what is meant by order and degree

**DEFINITION 1.1.**
(a) Order \( P/p = e \) \( = P^e | p, p^{e+1}/P \)
(b) Degree \( P/p = f = N_{k/k} P = p^f \)

**LEMMA 1.2.** Both order and degree are multiplicative
\[
\begin{align*}
\text{Order } P/p &= \text{order } P/P_L \cdot \text{order } P_L/p \\
\text{Degree } P/p &= \text{degree } P/P_L \cdot \text{degree } P_L/p
\end{align*}
\]

Let us assume here that \( K/k \) for \( [K:k] = n \) is a normal extension. This makes \( K/L \) normal for each \( L \) in the tower but not in \( L/k \). Let \( p \) have factors \( P_L^{(j)} \) in \( L \) for \( j = 1, 2, 3, \ldots, g \),
\[
p = \bigcap_{j=1}^{g} P_L^{(j)e}, \quad N\left(P_L^{(j)} \right)^f = N(p)^f
\] (1.2a)
Let order \( K/k \ P = e \) and degree \( K/k \ P = f \). Then for \( P = p \), we have order \( p = \text{degree } p = 1 \) from \( k \) to \( k \).

Thus from \( k \) to \( K \) the order has grown from 1 to \( e \) and the degree has grown from 1 to \( f \) and the number of factors in (1.2a) and (1.2b) has grown from 1 to \( g \). We arrange the tower fields in 1.1 in such a way that will separate the growths for \( K/k \) normal.

Let \( K_Z \) be a maximal \( L \) in \( \{ L : K \supseteq L \supseteq k \} \). \( K_Z \) is called the "splitting" field of \( P \) in \( K/L \) and is such that

\[
\text{degree } P_L/p = 1 \\
\text{order } P_L/p = 1
\]

Let us assume that \( K_T \) is a maximal \( L \) in \( \{ L : K \supseteq L \supseteq k \} \). \( K_T \) is called the "inertial" field of \( P \) in \( K/L \) and is such that

\[
\text{degree } P_L/p = f_L \geq 1 \\
\text{order } P_L/p = e_L \text{ for } (e_L, p) = 1.
\]

This maximality process can be performed again for all \( L \) such that:

\[
\text{degree } P_L/p = f_L \geq 1 \\
\text{order } P_L/p = e_L \text{ for } (e_L, p) = 1.
\]

The maximal field here is called the "first ramification" field \( K_{v_e} \).

For this field, \( F_L = f \) and \( e_L \) is a part of \( e \) prime to \( p \). This part is called "tame ramification." If order \( e \) is divisible by \( p \), the ramification is called "wild." Thus we have the new tower fields for extension \( K/k \):

\[
K \supseteq ... \supseteq K_{v_e} \supseteq K_T \supseteq K_Z \supseteq k \tag{1.2c}
\]

It is easier to define 1.2c by the Galois group methods.

**DEFINITION 1.2.** Let \( K/k \) be a normal extension. The Hilbert sequence for an ideal \( P \) in \( K \) is given by the subgroups of \( G = \text{Gal}(K/k) \) as follows:

\[
K \supseteq ... \supseteq K_{v_e} \supseteq K_T \supseteq K_Z \supseteq k \\
1 \subseteq ... \subseteq G_{v_e} \subseteq G_T \subseteq G_Z \subseteq G \	ag{1.3a}
\]

\[
k_{v_e} \overset{G}{\leftrightarrow} \{ u \in G : P^u = P \text{ or } A \equiv 0 \equiv A^{v_e} \equiv 0 \mod p \} = G_{v_e} \	ag{1.3b}
\]

\[
k_T \overset{G}{\leftrightarrow} \{ u \in G : P^u \equiv A \mod p \} = (G_{v_e}) \	ag{1.3c}
\]

\[
k_{v_e} \overset{G}{\leftrightarrow} \{ u \in G : A^{v_e} \equiv A \mod p^{r+1} \} = G_{v_e}, \ (r \geq 0). \	ag{1.3d}
\]

Where \( A \) is an arbitrary integer in \( O_k \). Since \( G_Z \) fixes \( P \), then \( G_T, G_{v_e} \), and so on are invariant subgroups of \( G_Z \). Since \( G_Z \) preserves \( P \), it is one of \( g \) conjugates,

\[
|G/G_Z| = g, \tag{1.3e}
\]

also, since \( G_T \) preserves each residue class mod \( P \),

\[
|G_Z/G_T| = |(O_K/P)/(O_k/p)| = |c(f)| = f, \tag{1.3f}
\]

which refer to the cyclic Galois group of an extension of a finite field. Furthermore

\[
|G_T| = e. \tag{1.3g}
\]
If \( r = e_0 p^w \), where \((e_0, p) = 1\), then there is a cyclic quotient,
\[
|G_T/G_{v_r}| = e_0
\]  
followed by future quotient groups of type \( C(p) \times C(p) \times \cdots \times C(p) \), with
\[
G_{v_r}/G_{v_{r+1}} = p^{w_r} (w_r \geq 0, \sum w_r = w).
\]  
(1.3i)

Here there is only a finite number \( w_r > 0 \), indeed \( p^w \mid n \)  
More general details of the above can be found in [1], [2], [3], [4], [5], [6], [7]

2. COMPUTING HILBERT SEQUENCE FOR \( K = k(\sqrt{d_1}, \sqrt{d_2}) \), FOR \( k = Q \).

Computing Hilbert sequence for \( K = k(\sqrt{d}) \), \( k = Q \), is contained in [1, p 89]. So we process to
\( K \supseteq K_i = Q(\sqrt{d_i}) \) for \( i = 1, 2, 3 \). Let \( d_3 = d_1 \cdot d_2/t^2 \) which means \( d_3 \) is square factor free, where \( d_i \) is the discriminant of \( k_i \).

Let \( G = \{1, u_1, u_2, u_3\} \), where \( u_i : \sqrt{d_i} \rightarrow \sqrt{d_i}, \sqrt{d_j} \rightarrow -\sqrt{d_j} \) for \( i \neq j \), then we have
\[
k_i = Q(\sqrt{d_i}) \cong G_i = \{1, u_i\}.
\]

Here we will build a tower of fields \( K \supseteq \cdots \supseteq K_{v_1} \supseteq K_T \supseteq K_Z \supseteq Q \) by using the Hilbert sequence in
Definition 1.2 for different types of primes \( p \) in \( Q \).

a Let \( p = P_1 \) (unramified) where the \( P_i \)'s are primes in \( K \) for \((d_1/p) = (d_2/p) = (d_3/p) = 1 \)  
where:
\[
(a/p) = \begin{cases} 
1 & \text{if } x^2 = a \mod p \text{ solvable for } x \text{ integer, } a|p \\
-1 & \text{if } x^2 \neq a \mod p \text{ for } x \text{ integer, } a|p \\
0 & \text{if } a|p.
\end{cases}
\]

Here \( f = e = 1 \) then \( g = 4 \) by 1.1. From \( |G/G_{Z}\| = g = 4 \) in (1.3e) we get that, \( |K_Z/k| = 4 \) and
\( K_Z = K \) and from \( |G_Z/G_T| = f = 1 \) in (1.3d), \( |K_T/K_Z| = 1 \) and so \( K_T = K \). Since
\( |G_T| = e = 1 \) in (1.3g), and from \( |G_T/G_{v_0}| = e_0 = 1 \) in (1.3h) and (1.3i) for \( r = 0, 1, 2, 3 \) then
\( |G_{v_r}/G_{v_{r+1}}| = |K_{v_{r+1}}/K_{v_r}| = 1 \)
\[K_{v_1} = K_{v_2} = K_{v_3} = K_{v_4} = K.\]

Thus, we have the following field tower for \( K/k : \)
\[k = Q \subseteq K_Z \subseteq K_T \subseteq K_{v_1} \subseteq K_{v_2} \subseteq K_{v_3} \subseteq K_{v_4} \subseteq K.\]

b Let \( p = P_1 P_2 \) (unramified) for \(- (d_1/p) = -(d_2/p) = (d_3/p) = 1 \). Here \( e_1 = e_2 = 1 \),
\( f_1 = f_2 = 2 \) and \( g = 2 \). Again from \( |G/G_{Z}| = g = 2 \) in (1.3e) we have: \( |K_Z/k| = 2 \) and by (1.3b)
\( K_Z = Q(\sqrt{d_3}) \). From \( |G_Z/G_T| = f = 1 \) then \( |K_Z/K_T| = 2 \) and then \( K_T = K \) Using the
same proof as above: \( K_{v_1} = K_{v_2} = K_{v_3} = K_{v_4} = K.\) This produces the following tower fields for
\( K/k : \)
\[k = Q \subseteq K_Z \subseteq K_T \subseteq K_{v_1} \subseteq K_{v_2} \subseteq K_{v_3} \subseteq K_{v_4} \subseteq K.\]

\[Q \subseteq K \subseteq K = K = K = K = K = K.\]

c \( p = P_1^2 \cdot P_2^2 \), where \( p \) is odd and \( p|d_1, p|d_2, p|d_3 \) and \( (d_3/p) = 1 \). Here \( e_1 = e_2 = 2 \) and
\( f_1 = f_2 = 1 \) and so \( g = 2 \). Since again \( |G/G_{Z}| = g = 2 \) then \( |K_Z/k| = 2 \) and by (1.3b)
\( K_Z = Q(\sqrt{d_3}) \). From \( |G_Z/G_T| = f = 1 \) then \( K_T = K_Z = k_3 = Q(\sqrt{d_3}) \) From
\( |G_T| = e = 2 = e_0 \cdot p^w = 1 \cdot 2^1 \) then by (1.3i) \( |G_{v_r}/G_{v_{r+1}}| = p^{w_r}r = 2^1 \) and from here for \( r = 0 : \)
\[ |G_{v_i}/G_{v_0}| = |K_{v_i}/K_{T}| = 2 \] and thus \( K_{v_i} = K \) and also \( K_{v_i} = K_{v_0} = K_{v_0} = K \), because
\[ |G_{v_i}/G_{v_{i+1}}| = |K_{v_{i+1}}/K_{v_i}| = 2^0 = 1 \] which produces the following tower fields for \( K/k \)
\[ k = Q \subseteq k_2 \subseteq k_T \subseteq k_{v_i} \subseteq k_{v_0} \subseteq k_{v_3} \subseteq k_{v_4} \subseteq K \]
\[ Q \subseteq k_3 \subseteq K = K = K = K = K. \]

d. \( p = P_1^2 \) for \( p \) odd, \( p|d_1, p|d_2, p|d_3, (d_3/p) = -1 \) with the same proof as above, the following
tower fields are produced.
\[ K_Z = Q, K_T = k_3, \text{ and } K_{v_1} = K_{v_0} = K_{v_3} = K_{v_4} = K. \]
e. \( P = p_1^2 p_2^2 \), and \( d_1 \equiv d_2 \equiv 1^2 \pmod{16}, d_3 \equiv 1 \pmod{8} \) produces the tower
\[ k = Q \subseteq k_2 \subseteq K_T \subseteq K_{v_1} \subseteq K_{v_2} \subseteq K_{v_3} \subseteq K_{v_4} \subseteq K \]
\[ Q = Q \subseteq k_3 \subseteq K = K = K = K. \]
f. \( p = p_1^2 \) for \( d_1 \equiv d_2 \equiv 12 \pmod{16}, d_3 \equiv 5 \pmod{8} \). Here \( e = 2 \) and \( g = 1 \) then \( f = 2 \). From
\[ |G/G_Z| = g = |K_Z/Q| = 1, K_Z = Q \] and by \[ |G_Z/G_T| = f = |K_T/K_Z| = 2, K_T \] is a quadratic
extension over \( Q \), then by (1.3c) \( K_T = k_3, e = 2 = e^0 \cdot p^w = 1.2^w \) and \( |G_{v_i}/G_{v_{i+1}}| = 2^w \) where \( \Sigma \)
\[ w_r = w \] and \( w_r \geq 0. \) From
\[ |G_{v_i}/G_{v_1}| = |K_{v_i}/K_T| = 2^0 = 1, K_{v_1} = k_3. \]
\[ |G_{v_i}/G_{v_1}| = 2^1 = |K_{v_i}/G_{v_1}| = 2, \text{ then } K_{v_2} = K, \text{ and with some proof } K_{v_3} = K_{v_2} = K_{v_4} = K \text{ producing} \]
\[ k = Q \subseteq K_Z \subseteq K_T \subseteq K_{v_1} \subseteq K_{v_2} \subseteq K_{v_3} \subseteq K_{v_4} \subseteq K \]
\[ Q = Q \subseteq k_3 \subseteq k_3 \subseteq K = K = K = K. \]
g. \( p = p_1^2 p_2^2 \) for \( d_1 \equiv d_2 \equiv 8 \pmod{16}, d_3 \equiv 1 \pmod{8} \) has the same tower fields as e.
h. \( p = p_1^2 \) for \( d_1 \equiv d_2 \equiv 8 \pmod{16}, d_3 \equiv 5 \pmod{8} \) also has the same Hilbert sequence as f.
i. \( p = p_1^4 \) for \( d_1 \equiv d_2 \equiv 8 \pmod{16}, d_3 \equiv 12 \pmod{8} \) has the following tower fields
\[ k = Q \subseteq k_2 \subseteq K_T \subseteq K_{v_1} \subseteq K_{v_2} \subseteq K_{v_3} \subseteq K_{v_4} \subseteq K \]
\[ Q = Q = Q = Q \subseteq k_3 = k_3 \subseteq K = K. \]

We showed in the above cases, if the prime ideal \( p \) of \( K \) does not split into \( n \) distinct prime factors of \( K \),
how we can build intermediate fields \( K_Z, K_T, K_{v_0}, \ldots \) where splitting of prime \( p \) occurs.

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REFERENCES