ON THE SEMI-INNER PRODUCT IN LOCALLY CONVEX SPACES

SHIH-SEN CHANG
YU-QING CHEN
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064, PEOPLE'S REPUBLIC OF CHINA

BYUNG SOO LEE
Department of Mathematics
Kyungsung University
Pusan 608-736, KOREA

(Received March 15, 1995 and in revised form June 20, 1995)

ABSTRACT. The purpose of this paper is to introduce the concept of semi-inner products in locally convex spaces and to give some basic properties

KEY WORDS AND PHRASES: Semi-inner product, duality mapping, upper semi-inner product, lower semi-inner product.

1991 AMS SUBJECT CLASSIFICATION CODES: 46C50

1. INTRODUCTION

The concept of semi-inner products in real normed spaces was first introduced by G. Lumer [6], but its history can be traced to S Mazur [8]. Recently, the semi-inner product theory has made great progress (cf. [9,11]) and it plays an important role in the theory of accretive operators and dissipative operators, differential equations, linear and nonlinear semigroups in Banach spaces and Banach space geometry theory (see [1,2,3,4,5,7]) The purpose of this paper is to introduce the concept of semi-inner products in locally convex spaces and to study their basic properties. As for the applications of our results, we shall give in another paper.

2. MAIN RESULTS

In this section, we shall always assume that $E$ is a real locally convex space generated by a family of seminorms $\{p_i\}_{i \in I}$, where $I$ is an index set

PROPOSITION 2.1. For each $x \in E$, $y \in E$ and $i \in I$, the following hold:

(i) $h^{-1}(p_i(x + hy) - p_i(x))$ is a nondecreasing function in $h \in (0, + \infty)$ and it is bounded from below,

(ii) $h^{-1}(p_i(x) - p_i(x - hy))$ is nonincreasing in $h \in (0, + \infty)$ and bounded from upper,

(iii) $h^{-1}(p_i(x) - p_i(x - hy)) \leq h^{-1}(p_i(x + hy) - p_i(x))$ for $h \in (0, + \infty)$

PROOF. (i) For any $h_1, h_2 \in (0, + \infty)$, $h_1 < h_2$, since

\[
p_i(x + h_1 y) - p_i(x) = p_i(x + h_2 h_2^{-1} h_1 y) - p_i(x)
\]

\[
= p_i((h_1 h_2^{-1})(x + h_2 y) + (1 - h_1 h_2^{-1})x) - p_i(x)
\]

\[
\leq p_i((h_1 h_2^{-1})(x + h_2 y)) + p_i((1 - h_1 h_2^{-1})x) - p_i(x)
\]

\[
= h_1 h_2^{-1} p_i(x + h_2 y) + (1 - h_1 h_2^{-1})p_i(x) - p_i(x)
\]

\[
= h_2^{-1} h_1 (p_i(x + h_2 y) - p_i(x)).
\]
Therefore we have \( h_1^{-1}(p_1(x + h_1 y) - p_1(x)) \leq h_2^{-1}(p_1(x + h_2 y) - p_1(x)) \).

Moreover, it is obvious that \( h_1^{-1}(p_1(x + h y) - p_1(x)) \geq p_1(y) \).

(i) By the same way, we can prove that (ii) is true.

(ii) is obvious.

Next, we define

\[
[x, y]_i^+ = \lim_{h \to 0^+} h^{-1}(p_1(x + hy) - p_1(x)), \\
[x, y]_i^- = \lim_{h \to 0^+} h^{-1}(p_1(x) - p_1(x - hy)).
\]

Now we list some properties of \([x, y]_i^\pm\) as follows:

**PROPOSITION 2.2.** (i) \([x, y]_i^+ \leq [x, y]_i^+\),

(ii) \(|[x, y]_i^\pm| \leq p_1(y),

(iii) \([x, y]_i^\pm - [x, z]_i^\pm \leq p_1(y - z),

(iv) \([x, y]_i^+ = -[x, -y]_i^+ = -[x, y]_i^-\),

(v) \([x, y]_i^\pm = s[x, y]_i^\pm, \ r, s \geq 0\),

(vi) \([x, y + z]_i^+ \leq [x, y]_i^+ + [x, z]_i^+\) and \([x, y + z]_i^- \geq [x, y]_i^- + [x, z]_i^-\);

(vii) \([x, y + z]_i^+ \geq [x, y]_i^+ + [x, z]_i^-\) and \([x, y + z]_i^- \leq [x, y]_i^- + [x, z]_i^+\);

(viii) \([x, y + ax]_i^\pm = [x, y]_i^\pm + \alpha p_1(x), \ \forall \alpha \in \mathbb{R}\),

(ix) \([x, y]_i^-\) is upper semi-continuous in \(x, y \in E\) and \([x, y]_i^+\) is lower semi-continuous in \(x, y \in E\);

(x) If \(x(t) : [a, b] \to E\) is differentiable in \(t \in (a, b)\) in the sense that

\[
\lim_{\Delta t \to 0} \frac{p_1(x(t + \Delta t) - x(t) - x'(t) \Delta t)}{\Delta t} = 0 \quad \text{for all} \quad i \in I
\]

and \(m_i(t) = p_i(x(t))\), then

\[
D^+ m_i(t) = \lim_{h \to 0^+} \frac{m_i(t + h) - m_i(t)}{h} = [x(t), x'(t)]_i^+,
\]

\[
D^- m_i(t) = \lim_{h \to 0^+} \frac{m_i(t) - m_i(t - h)}{h} = [x(t), x'(t)]_i^- , \quad i \in I.
\]

**PROOF.** (i)-(v) is obvious.

(vi) Since

\[
\frac{h^{-1}(p_1(x + hy + z)) - p_1(x)}{h} = \frac{1}{h} \left( \int (x + hy + z) - p_1(x) \right) - \int (x + 2hy + z) - p_1(x) \right) \leq \frac{1}{h} \left( \int (x + 2hy) - p_1(x) \right) - \int (x + 2hx) - p_1(x) \right),
\]

we know that \([x, y + z]_i^+ \leq [x, y]_i^+ + [x, z]_i^+\). On the other hand, since

\[
\frac{h^{-1}(p_1(x) - p_1(x - hy + z))}{h} = \frac{1}{h} \left( \int (x - hy + z) - p_1(x) \right) - \int (x - 2hy + z) - p_1(x) \right),
\]

by the same way we can prove that

\[
[x, y + z]_i^- \geq [x, y]_i^- + [x, z]_i^-.
\]

(vii) By (vi) \([x, y]_i^+ = [x, y + z - z]_i^+ \leq [x, y + z]_i^+ + [x, -z]_i^+\). By (iv), \([x, -z]_i^+ = -[x, z]_i^-\), and so \([x, y]_i^+ + [x, z]_i^- \leq [x, y + z]_i^+\). By (vi) and (iv) again, we have \([x, y + z]_i^- \leq [x, y]_i^- + [x, z]_i^-\).

(viii) Since \([x, y + ax]_i^+ \leq [x, y]_i^+ + [x, ax]_i^+ = [x, y]_i^+ + \alpha p_1(x)\), by (vii) we have \([x, y + ax]_i^- \geq [x, y]_i^- + [x, ax]_i^+ = [x, y]_i^- + \alpha p_1(x)\), and so \([x, y + ax]_i^- = [x, y]_i^- + \alpha p_1(x)\).

Similarly we can prove that \([x, y + ax]_i^- = [x, y]_i^- + \alpha p_1(x)\).

(ix) Since
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Given $x, y \to x, y$, we get
\[
\lim_{h \to 0} [x, y]^+ \leq \lim_{h \to 0} h^{-1}(p_i(x + hy) - p_i(x)) = h^{-1}(p_i(x + hy) - p_i(x)),
\]
and so
\[
\lim_{h \to 0} [x, y]^+ \leq \lim_{h \to 0} h^{-1}(p_i(x + hy) - p_i(x)) = [x, y]^+.
\]

On the other hand, since $[x, y]^+ \geq h^{-1}(p_i(x) - p_i(x - hy))$, we have
\[
\lim_{h \to 0} [x, y]^+ \geq [x, y]^+.
\]

(x) Since
\[
\left| h^{-1}(p_i(x(t) + hz(t)) - p_i(x(t))) \right| \leq h^{-1}(p_i(x(t) + hz(t)) - p_i(x(t))),
\]
as $h \to 0^+$, we know that $D^+ = [x(t), x(t)]^+$. Similarly we can prove that $D^- = [x(t), x(t)]^-$. Let $E^*$ be the dual space of $E$. For each $i \in I$ we define a mapping $j_i : E \to 2^{E^*}$ by
\[
j_i(x) = \{f_i \in E^* : f_i(x) = p_i(x) \text{ and } [x, y]^+ \leq f_i(y) \leq [x, y]^+ \quad \forall y \in E\}.
\]

It is obvious that $j_i(x)$ is convex. Next we prove that $j_i(x) \neq \emptyset$ for each $x \in E$. In fact, for any given $y_0 \in E$, $y_0 \neq 0$ we define
\[
f_i(\alpha y_0) = \alpha [x, y_0]^+.
\]

1. If $\alpha \geq 0$, then $f_i(\alpha y_0) = [x, \alpha y_0]^+$,
2. If $\alpha < 0$, then $f_i(\alpha y_0) = -\alpha [x, y_0]^+ = [-x, -\alpha y_0]^+ = [x, -\alpha y_0]^+ = [x, y_0]^-$.

Hence we have $f_i(\alpha y_0) \leq [x, \alpha y_0]^+$ for all $\alpha \in \mathbb{R}$. By Proposition 2.2, $[x, y]^+$ is a subadditive function of $y \in E$ by Hahn-Banach theorem [10], there exists a linear function $\tilde{j}_i : E \to \mathbb{R}$ such that $\tilde{j}_i(\alpha y_0) = f_i(\alpha y_0)$ for all $\alpha \in \mathbb{R}$ and $[x, y]^+ \leq \tilde{j}_i(y) \leq [x, y]^+$, $\forall y \in E$, i.e., $[x, y]^+ \leq \tilde{j}_i(y) \leq [x, y]^+$, $|\tilde{j}_i(y)| = p_i(y)$.

This implies that $\tilde{j}_i \in j_i(x)$.

By the above argument and the Banach-Alaoglu theorem (see [10]) we have the following.

**Proposition 2.3.** For any $x \in E$, $i \in I$, $j_i(x)$ is a nonempty weak* compact convex subset of $E^*$.

**Proposition 2.4.** $[x, y]^+ = \max\{f_i(y) : f_i \in j_i(x)\}$

**Definition 2.1.** For each $i \in I$, $(x, y)^+ = p_i(x) \cdot [x, y]^+$ is called the upper semi-inner product with respect to $i \in I$. $(x, y)^- = p_i(x) \cdot [x, y]^- \cdot [y, x]^-$ is called the lower semi-inner product with respect to $i \in I$.

**Definition 2.1.** For any $i \in I$, we define the mapping $j_i : E \to 2^{E^*}$ by
\[
j_i(x) = p_i(x) \cdot j_i(x) \quad \text{for all } x \in E,
\]
and it is called the duality mapping with respect to \(i \in I\).

The following results can be obtained from Proposition 2.2-2.4 immediately.

**PROPOSITION 2.5.** The semi-inner product defined in Definition 2.1 has the following properties:

(i) \((x, y)^+_i \leq (x, y)_i^+\),

(ii) \(|(x, y)|^+_i \leq p_i(x) \cdot p_i(y)\),

(iii) \(|(x, y)|^+_i - (x, z)^+_i| \leq p_i(x) \cdot p_i(y - z)\),

(iv) \((x, y)^-_i = -(x, -y)^+_i = -(x, y)_i^-\),

(v) \((s, r) = s \cdot (x, y)^+_i, r, s \geq 0\),

(vi) \(\langle x, y + z \rangle^+_i \leq \langle x, y \rangle^+_i + \langle x, z \rangle^+_i\) and \(\langle x, y + z \rangle_i^- \geq \langle x, y \rangle_i^- + \langle x, z \rangle_i^-\);

(vii) \(\langle x, y + z \rangle_i^+ \geq \langle x, y \rangle_i^+ + \langle x, z \rangle_i^+\) and \(\langle x, y + z \rangle_i^- \leq \langle x, y \rangle_i^- + \langle x, z \rangle_i^-\);

(viii) \((x, y + \alpha z)^+_i = (x, y)^+_i + \alpha p_i(z)^+\), \(\forall \alpha \in \mathbb{R}\);

(ix) \((x, y)^+_i\) is upper semi-continuous and \((x, y)_i^-\) is lower semi-continuous;

(x) If \(x(t) \in [a, b] \in I\) is differentiable in \(t \in (a, b)\) in the sense that

\[
\lim_{\Delta t \to 0} \frac{p_i(x(t + \Delta t) - x(t) - x'(t) \Delta t)}{\Delta t} = 0, \quad \forall t \in I,
\]

and \(m_i(t) = p_i^2(x(t)), \) then

\[
D^+ m_i(t) = 2(x(t), x'(t))^+_i \quad \text{and} \quad D^+ m_i(t) = 2(x(t), x'(t))^+_i.
\]

**PROPOSITION 2.6.** For any \(i \in I, x \in E, J_i(x)\) is nonempty, weak* compact convex, and

\[
\langle x, y \rangle^+_i = \max \{f_i(y) : f_i \in J_i(x)\}
\]

\[
\langle x, y \rangle^-_i = \min \{f_i(y) : f_i \in J_i(x)\}.
\]

**DEFINITION 2.3.** Let \( \phi : E \to \mathbb{R} \) be any given convex function. The subdifferential of \( \phi \) at \( x \in E \) (denoted by \( \partial \phi(x) \)) is defined by

\[
\partial \phi(x) = \{ f \in E^* : \phi(x) - \phi(y) \leq f(x - y) \quad \text{for all} \quad y \in E \}\.
\]

**THEOREM 2.1.** Let \( \phi_i(x) = \frac{1}{2} p_i^2(x), x \in E, \) then the subdifferential \( \partial \phi_i \) is identical to duality mapping \( J_i \).

**PROOF.** Let \( f \in J_i(x) \), then by (2.1) and Definition 2.2 and the fact that \(|(x, y)^+_i| \leq p_i(y)\), we have

\[
f(x - y) = f(x) - f(y) \geq p_i^2(x) - p_i(x) \cdot p_i(y) \geq \frac{1}{2} (p_i^2(x) - p_i^2(y)),
\]

and so, \( f \in \partial \phi_i(x) \).

Conversely, if \( f \in \partial \phi_i(x) \), then

\[
p_i^2(x) \leq p_i^2(y) + 2 \cdot f(x - y) \quad \text{for all} \quad y \in E. \quad (2.2)
\]

Replacing \( y \) by \( x + hy \) in (2.2) we have

\[
p_i^2(x) \leq p_i^2(x + hy) - 2h \cdot f(y) \quad \text{for all} \quad y \in E \quad \text{and} \quad h \in \mathbb{R}. \quad (2.3)
\]

When \( h > 0 \), we have

\[
\frac{1}{2} (p_i(x + hy) + p_i(x)) \cdot \frac{1}{h} (p_i(x + hy) - p_i(x)) \geq f(y), \quad \forall y \in E. \quad (2.4)
\]

Letting \( h \to 0^+ \) we have

\[
p_i(x) \cdot [x, y]^+_i \geq f(y), \quad \forall y \in E. \quad (2.5)
\]

If \( p_i(x) = 0 \), then \( f = 0 \). Therefore \( f \in p_i(x)j_i(x) = J_i(x) \), the desired conclusion is proved. If \( p_i(x) \neq 0 \), for \( h < 0 \), we have
\[ f(y) \geq \frac{1}{2} (p_i(x + h y) + p_i(x)) + \frac{1}{h} (p_i(x + h y) - p_i(x)), \quad \forall h < 0, \ y \in E. \]

Letting \( h \to 0^- \), we have

\[ f(y) \geq p_i(x) \cdot [x, y]_i^+. \quad (2.6) \]

By (2.5) and (2.6), we know that \( \frac{f}{p_i(x)} \in j_i(x) \), i.e., \( f \in p_i(x) \cdot j_i(x) = J_i(x) \)

This completes the proof.

**Definition 2.4.** Let \( A : D(A) \subseteq E \to 2^E \) be a nonlinear multi-valued mapping. \( A \) is said to be accretive, if

\[ p_i(x - y) \leq p_i(x - y + \lambda(u - v)) \]

for all \( x, y \in D(A), u \in A(x), v \in A(y), i \in I, \lambda > 0 \).

**Theorem 2.2.** The following conclusions are equivalent:

(i) \( A : D(A) \subseteq E \to 2^E \) is accretive,

(ii) \( [x - y, u - v]_i^+ \geq 0 \) for all \( x, y \in D(A), u \in Ax, v \in Ay, i \in I \),

(iii) \( (x - y, u - v)_i^+ \geq 0 \) for all \( x, y \in D(A), u \in Ax, v \in Ay, i \in I \)

**Proof.** (i) \( \Rightarrow \) (ii). Since \( \lambda^{-1}(p_i(x - y + \lambda(u - v)) - p_i(x - y)) \geq 0 \), let \( \lambda \to 0^+ \) we get (i).

(ii) \( \Rightarrow \) (iii). Since \( (x - y, u - v)_i^+ = p_i(x - y)[x - y, u - v]_i^+ \).

(a) If \( p_i(x - y) = 0 \), then \( \lambda^{-1}(p_i(x - y + \lambda(u - v))) \geq 0 \), and so \( [x - y, u - v]_i^+ \geq 0 \).

(b) If \( p_i(x - y) \neq 0 \), then \( [x - y, u - v]_i^+ \geq 0 \).

(iii) \( \Rightarrow \) (i). By Proposition 2.1, \( \lambda^{-1}(p_i(x - y + \lambda(u - v)) - p_i(x - y)) \) is nondecreasing in \( \lambda \in (0, + \infty) \) and

\[
\lim_{\lambda \to 0^+} \frac{p_i(x - y + \lambda(u - v)) - p_i(x - y) - [x - y, u - v]_i^+}{\lambda} \geq 0.
\]

This completes the proof.

**Theorem 2.3.** Let \( A : D(A) \subseteq E \to 2^E \) be an accretive mapping and \( x : [0, + \infty) \to E \) be continuous. If the following conditions are satisfied:

(i) there exists \( x'(t) : [0, + \infty) \to E \) such that

\[
\lim_{\Delta t \to 0^+} \frac{p_i(x(t + \Delta t) - z(t) - x'(t) \Delta t)}{\Delta t} = 0, \quad \forall t \in I;
\]

(ii) \( x(0) = x_0 \in D(A) \);

(iii) \( x'(t) \in -Ax(t) \) a.e. \( t \in (0, + \infty) \),

then such an \( x(t) \) is unique.

**Proof.** Suppose the contrary, there exists another \( y : [0, + \infty) \to E \) which is continuous and satisfies conditions (i)-(iii). Let \( m_i(t) = p_i(z(t) - y(t)) \). By (X) in Proposition 2.2, we know that

\[ D^-m_i(t) = [x(t) - y(t), \lambda(t) - y(t)]_i^- \, . \]

Furthermore, there exist \( u(t) \in Ax(t) \) and \( v(t) \in Ay(t) \) such that \( x'(t) = u(t), y'(t) = v(t) \) a.e. \( t \in (0, + \infty) \), hence we have

\[ D^-m_i(t) = [x(t) - y(t), -u(t) + v(t)]_i^- \, . \]

It follows from Theorem 2.2 that \( D^-m_i(t) \leq 0 \), and so

\[ p_i(z(t) - y(t)) \leq p_i(z(0) - y(0)) = 0 \quad \text{for all} \quad i \in I. \]

This implies that \( x(t) = y(t) \) for all \( t \in [0, + \infty) \).
THEOREM 2.4. Let $M \subset E$ be a nonempty convex subset and $x \in E$ be a given point. Then the following conditions are equivalent:

(i) $p_i(y_0 - x) \leq p_i(y - x)$ for all $y \in M$,

(ii) $[y_0 - x, y - y_0]^+ \geq 0$

PROOF. (i) $\Rightarrow$ (ii) Since $p_i(y_0 - x) \leq p_i(y - x)$ for all $y \in M$, letting $z = y_0 + (1 - \alpha)(y - y_0)$ for any $y \in M$, $\alpha \in (0, 1)$, then $z \in M$ (since $M$ is convex), and so $p_i(y_0 - x) \leq p_i(y_0 - x + (1 - \alpha)(y - y_0))$, $\alpha \in (0, 1)$, $y \in M$,

$$i.e., \quad \frac{p_i((y_0 - x) + (1 - \alpha)(y - y_0)) - p_i(y_0 - x)}{1 - \alpha} \geq 0, \quad \forall y \in M, \quad \alpha \in (0, 1).$$

Letting $\alpha \to 1$ we get

$$[y_0 - x, y - y_0]^+ \geq 0 \quad \text{for all} \quad y \in M.$$

(ii) $\Rightarrow$ (i) Since $[y_0 - x, y - y_0]^+ \geq 0$, we have

$$\frac{1}{h} (p_i((y_0 - x) + h(y - y_0)) - p_i(y_0 - x)) \geq 0, \quad \forall h > 0,$$

i.e., $p_i(y_0 - x) \leq p_i(y_0 - x + h(y - y_0))$, $\forall h > 0$. Letting $h \to 1$ we have

$$p_i(y_0 - x) \leq p_i(y - x) \quad \text{for all} \quad y \in M.$$

This completes the proof.

ACKNOWLEDGMENT. The first author was supported by the National Natural Science Foundation of China and the third author was supported in part by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1994.

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