A FIXED POINT THEOREM FOR NON-SELF SET-VALUED MAPPINGS

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ABSTRACT. Let X be a complete, metrically convex metric space, K a closed convex subset of X, CB(X) the set of closed and bounded subsets of X. Let F: K → CB(X) satisfying definition (1) below, with the added condition that Fx ⊆ K for each x ∈ ∂K. Then F has a fixed point in K. This result is an extension to multivalued mappings of a result of Ćirić [1].

KEY WORDS AND PHRASES. Fixed point, multivalued map, non-self map

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Let X be a complete metrically convex metric space. This means that, for each x, y in X, x ≠ y, there exists a z in ∂X such that d(x, y) = d(x, z) + d(z, y). Let CB(X) denote the set of closed and bounded subsets of X, H denote the Hausdorff metric on CB(X). Let K be a nonempty closed, convex subset of X.

Let F: K → CB(X) satisfying: for each x, y in K,
\[ H(Fx, Fy) \leq h \max \left\{ \frac{d(x, y)}{a}, D(x, Fx), D(y, Fy), \frac{[D(x, Fy) + D(y, Fx)]}{a + h} \right\}, \]
(1)
where 0 < h < (−1 + √5)/2, a ≥ 1 + (2h²/(1 + h)), and F(x) ⊆ K for each x ∈ ∂K.

Ćirić [1] proved a fixed point theorem for the single-valued version of (1). He also established a multivalued version. However, he used the δ-distance, instead of the Hausdorff distance, so that the result and proof are identical to the single-valued case. It is the purpose of this paper to prove a multivalued version. For the single-valued version of (1), one can allow h to satisfy 0 < h < 1. However, the multivalued proof requires smaller values of h.

THEOREM. Let X be a complete metrically convex metric space, K a nonempty closed, convex subset of X. Let F: K → CB(X) satisfying (1), and the condition that Fx ⊆ K for each x ∈ ∂K. Then F has a fixed point in K.

PROOF. We shall need the following lemma of Nadler [2].

LEMMA. Let A, B ∈ CB(X), x ∈ A. Then, for each positive number α, there exists a y ∈ B such that
\[ d(x, y) \leq H(A, B) + \alpha. \]

We shall assign c h(1 + h). We shall now construct a sequence \{x_n\} in K in the following way. Let x_0 ∈ K and define x_1' ∈ Fx_0. If x_1' ∈ K, set x_1 = x_1'. If not, then select a point x_1 ∈ ∂K such that d(x_0, x_1) + d(x_1, x_1') = d(x_0, x_1'). Then x_1 ∈ K. By the Lemma, choose x_2' ∈ Fx_1 such that d(x_1', x_2') ≤ H(Fx_0, Fx_1) + α. If x_2' ∈ K, set x_2 = x_2'. Otherwise, choose x_2 so...
that \( d(x_1,x_2) + d(x_2,x'_2) = d(x_1,x'_2) \). By induction we obtain sequences \( \{x_n\}, \{x'_n\} \) such that, for \( n = 1, 2, \ldots \),

(i) \( x_{n+1}' \in Fx_n \),

(ii) \( d(x_{n+1}',x_n') \leq H(Fx_n,x_{n-1}) + \alpha^n \),

where

(iii) \( x_{n+1}' = x_{n+1} \) if \( x_{n+1}' \in K \), or

(iv) \( d(x_n,x_{n+1}) + d(x_{n+1},x_{n+1}') = d(x_n,x_{n+1}') \) if \( x_{n+1}' \notin K \) and \( x_{n+1} \in \partial K \).

Now define

\[ P := \{ x_i \in \{ x_n \} : x_i = x_i', \ i = 1, 2, \ldots \} ; \]
\[ Q := \{ x_i \in \{ x_n \} : x_i \neq x_i', \ i = 1, 2, \ldots \} . \]

Note that, if \( x_n \in Q \), for some \( n \), then \( x_{n-1} \in P \).

For \( n \geq 2 \) we shall consider \( d(x_n,x_{n+1}) \). There are three possibilities.

Case 1. \( x_n, x_{n+1} \in P \). Then, from (1),

\[
\begin{align*}
d(x_n,x_{n+1}) & = d(x_n',x_{n+1}') \\
& \leq H(Fx_{n-1},Fx_n) + \alpha^n \\
& \leq h \max \left\{ \frac{d(x_{n-1},x_n)}{a}, D(x_{n-1},Fx_{n-1}), D(x_n,Fx_n), \frac{D(x_{n-1},Fx_n) + D(x_n,Fx_{n-1})}{a + h} \right\} + \alpha^n \\
& \leq h \max \left\{ \frac{d(x_{n-1},x_n)}{a}, d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1}) + d(x_n,x_n)}{a + h} \right\} + \alpha^n \\
& \leq \max \left\{ hd(x_{n-1},x_n) + \alpha^n, \frac{\alpha^n}{1-h}, \frac{hd(x_{n-1},x_n) + \alpha^n(a+h)}{a} \right\} \\
& \leq hd(x_{n-1},x_n) + \max \left\{ \frac{1}{1-h}, \frac{a+h}{a} \right\} \alpha^n = hd(x_{n-1},x_n) + \frac{\alpha^n}{1-h}. \quad (2)
\end{align*}
\]

Case 2. \( x_n \in P, x_{n+1} \in Q \). Then, from (1),

\[
\begin{align*}
d(x_n,x_{n+1}) \leq d(x_n,x_{n+1}') & \leq H(Fx_{n-1},Fx_n) + \alpha^n \\
& \leq h \max \left\{ \frac{d(x_{n-1},x_n)}{a}, D(x_{n-1},Fx_{n-1}), D(x_n,Fx_n), \frac{D(x_{n-1},Fx_n) + D(x_n,Fx_{n-1})}{a + h} \right\} + \alpha^n \\
& \leq h \max \left\{ \frac{d(x_{n-1},x_n)}{a}, d(x_{n-1},x_n'), d(x_n,x_n'), \frac{d(x_{n-1},x_{n+1}) + d(x_n,x_n)}{a + h} \right\} + \alpha^n \\
& \leq \max \left\{ hd(x_{n-1},x_n) + \alpha^n, \frac{\alpha^n}{1-h}, \frac{hd(x_{n-1},x_n) + \alpha^n(a+h)}{a} \right\} \\
& \leq hd(x_{n-1},x_n) + \max \left\{ \frac{1}{1-h}, \frac{a+h}{a} \right\} \alpha^n = hd(x_{n-1},x_n) + \frac{\alpha^n}{1-h}. \quad (3)
\end{align*}
\]

Case 3. \( x_n \in Q, x_{n+1} \in P \). Note, that \( x_n \in Q \) implies that \( x_{n-1} \in P \). Using the convexity of \( X \),

\[ d(x_n,x_{n+1}) \leq \max \{ d(x_{n-1},x_{n+1}), d(x_n',x_{n+1}) \} \quad (4) \]

Suppose that the maximum of the right hand side of (4) is \( d(x_n',x_{n+1}) \). Then, from (1),

\[
\begin{align*}
d(x_n,x_{n+1}) \leq d(x_n',x_{n+1}') & \leq H(Fx_{n-1},Fx_n) + \alpha^n \\
& \leq h \max \left\{ \frac{d(x_{n-1},x_n)}{a}, D(x_{n-1},Fx_{n-1}), D(x_n,Fx_n), \frac{D(x_{n-1},Fx_n) + D(x_n,Fx_{n-1})}{a + h} \right\} + \alpha^n \\
& \leq h \max \left\{ \frac{d(x_{n-1},x_n)}{a}, d(x_{n-1},x_n'), d(x_n,x_n'), \frac{d(x_{n-1},x_{n+1}) + d(x_n,x_{n+1})}{a + h} \right\} + \alpha^n
\end{align*}
\]
Recall that \( d(x_{n-1}, x_n) \leq d(x_{n-1}, x'_n) \) and that \( d(x_n, x'_n) \leq d(x_{n-1}, x_n) \). Also, 
\[
d(x_{n-1}, x_{n+1}) + d(x'_n, x'_n) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x'_n, x_n) = d(x_{n-1}, x'_n) + d(x_n, x_{n+1}).
\]
Therefore,
\[
d(x_n, x_{n+1}) \leq h \max \left\{ d(x_{n-1}, x'_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x'_n) + d(x_n, x_{n+1})}{a + h} \right\} + \alpha^n
\]
\[
\leq \max \left\{ hd(x_{n-1}, x'_n) + \alpha^n, \frac{\alpha^n}{1 - h}, \frac{hd(x_{n-1}, x'_n) + \alpha^n(a + h)}{a} \right\}
\]
\[
\leq hd(x_{n-1}, x'_n) + \frac{\alpha^n}{1 - h}.
\]
Since \( x_{n-1} \in P \) and \( x_n \in Q \), it follows from Case 2, that
\[
d(x_n, x_{n+1}) \leq h^2 d(x_{n-2}, x_{n-1}) + \frac{h\alpha^{n-1}}{1 - h} + \frac{\alpha^n}{1 - h}.
\] (5)
If the maximum of the right hand side of (4) is \( d(x_{n-1}, x_{n+1}) \), then, from (1),
\[
d(x_n, x_{n+1}) \leq d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x'_n) + d(x'_n, x_{n+1})
\] (6)
\[
\leq d(x_{n-1}, x'_n) + H(Fx_{n-1}, Fx_n) + \alpha^n
\]
\[
\leq d(x_{n-1}, x'_n) + h \max \left\{ \frac{d(x_{n-1}, x_n)}{a}, D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n), \frac{[D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})]/(a + h)]}{a + h} \right\} + \alpha^n
\]
\[
\leq d(x_{n-1}, x'_n) + h \max \left\{ \frac{d(x_{n-1}, x_n)}{a}, d(x_{n-1}, x'_n), d(x_n, x_{n+1}), \frac{[d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)]/(a + h)}{a + h} \right\} + \alpha^n
\]
\[
\leq \max \left\{ (1 + h)d(x_{n-1}, x'_n) + \alpha^n, \frac{\alpha^n}{1 - h}, \frac{[d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)]/(a + h)}{a + h} \right\} + \alpha^n.
\]
Using (6), if the maximum of the quantity in braces is the third term, then
\[
d(x_{n-1}, x_{n+1}) \leq \frac{hd(x_{n-1}, x'_n) + (a + h)\alpha^n}{a} \leq \frac{hd(x_{n-1}, x'_n) + (a + h)\alpha^n}{a}.
\]
Therefore, by Case 2,
\[
d(x_n, x_{n+1}) \leq \max \left\{ (1 + h)d(x_{n-1}, x'_n) + \alpha^n, \frac{\alpha^n}{1 - h}, \frac{hd(x_{n-1}, x'_n) + (a + h)\alpha^n}{a} \right\}
\]
\[
\leq (1 + h)d(x_{n-1}, x'_n) + \frac{\alpha^n}{1 - h}
\]
\[
\leq h(1 + h)d(x_{n-2}, x_{n-1}) + \frac{h\alpha^{n-1}}{1 - h} + \frac{\alpha^n}{1 - h}.
\] (7)
Define \( \delta = \alpha^{-1/2} \max \{d(x_0, x_1) d(x_1, x_2)\} \). We shall now show that
\[
d(x_n, x_{n+1}) \leq \alpha^{n/2}(\delta + 3n), \quad n > 1.
\] (8)
The proof is by induction. Note that, for \( 0 \leq h < (-1 + \sqrt{5})/2, (1 + h)/(1 - h) < 3, \) and \( 1/(1 - h) < 3 \).
If \( x_2 \) and \( x_3 \) are such that (3) or (4) is satisfied, then
\[
d(x_2, x_3) \leq hd(x_1, x_2) + \frac{\alpha^2}{1 - h} \leq h\alpha^{1/2} \delta + 3\alpha^2 < \alpha(\delta + 3),
\]
since \( h < h(1 + h) = \alpha \).
Note that (5) implies (7). If $x_2$ and $x_3$ are such that (7) is satisfied, then
\[
d(x_2, x_3) \leq h(1 + h)d(x_1, x_2) + \frac{(1 + h)\alpha}{1 - h} + \frac{\alpha^2}{1 - h} \leq \alpha^{3/2}\delta + 3\alpha + 3\alpha^2 \leq \alpha(\delta + 6).
\]
Therefore, in all cases, $d(x_2, x_3) \leq \alpha(\delta + 6)$. Assume the induction hypothesis. If (3) or (4) are satisfied, then
\[
d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) + \frac{\alpha^n}{1 - h} \leq h\alpha^{(n-1)/2}(\delta + 3(n - 1)) + 3\alpha^n \\
\leq \alpha^{n/2}(\delta + 3n)
\]
If (7) is satisfied, then
\[
d(x_n, x_{n+1}) \leq h(1 + h)d(x_{n-2}, x_{n-1}) + \frac{(1 + h)\alpha^{n-1}}{1 - h} + \frac{\alpha^n}{1 - h} \\
\leq \alpha^{n/2}(\delta + 3(n - 2)) + 3\alpha^{n-1} + 3\alpha^n \leq \alpha^{n/2}(\delta + 3n).
\]
From (8) it follows that, for $m > n$,
\[
d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \delta \sum_{i=n}^{m-1} \alpha^{i/2} + 3 \sum_{i=n}^{m-1} \alpha^{i/2},
\]
and \{x_n\} is Cauchy, hence convergent. Call the limit $p$.

Let \{x_{n_k}\} denote the subsequence of \{x_n\} with the property that each term of the subsequence belongs to $P$. Then
\[
H(Fx_{n_k}, Fp) \leq h \max\{d(x_{n_k-1}, p)/a, D(x_{n_k-1}, Fx_{n_k-1}), D(p, Fp), \}
\]
\[
[D(x_{n_k-1}, Fp) + D(p, Fx_{n_k-1})]/(a + h)]
\]
\[
\leq h \max\{d(x_{n_k-1}, p)/a, d(x_{n_k-1}, x_{n_k}), D(p, Fp), \}
\]
\[
[D(x_{n_k-1}, Fp) + d(p, x_{n_k})]/(a + h)\}.
\]
Taking the limit as $k \to \infty$ yields
\[
H(p, Fp) \leq hD(p, Fp),
\]
which implies, since $H(p, Fp) = D(p, Fp)$, that $p \in Fp$.

REFERENCES

