RESEARCH NOTES
NOTES ON \((\alpha, \beta)\)-DERIVATIONS

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ABSTRACT. Let \( R \) be a prime ring of characteristic not 2, \( U \) a nonzero ideal of \( R \) and \( 0 \neq d \) a \((\alpha, \beta)\)-derivation of \( R \) where \( \alpha \) and \( \beta \) are automorphisms of \( R \). i) \([d(U), a] = 0\) then \( a \in Z \) ii) For \( a, b \in R \), the following conditions are equivalent (I) \( \alpha(a)d(x) = d(x)\beta(b) \), for all \( x \in U \) (II) Either \( \alpha(a) = \beta(b) \in C_R(d(U)) \) or \( C_R(a) = C_R(b) = R' \) and \( a[a, x] = [a, x]b \) (or \( a[b, x] = [b, x]b \)) for all \( x \in U \) Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a nonzero ideal of \( R \) iii) Let \( d \) be a \((\alpha, \beta)\)-derivation of \( R \) and \( g \) be a \((\gamma, \delta)\)-derivation of \( R \). Suppose that \( d g \) is a \((\alpha \gamma, \beta \delta)\)-derivation and \( g \) commutes both \( \gamma \) and \( \delta \) then \( g(x)U = 0 \) for all \( x, y \in U \). iv) Let \( \text{Ann}(U) = 0 \) and \( d \) be an \((\alpha, \beta)\)-derivation of \( R \) and \( g \) be a \((\gamma, \delta)\)-derivation of \( R \) such that \( g \) commutes both \( \gamma \) and \( \delta \). If for all \( x, y \in U \), \( \beta^{-1}(d(x))Ug(y) = 0 = g(x)U\alpha^{-1}(d(y)) \) then \( dg \) is a \((\alpha \gamma, \beta \delta)\)-derivation on \( R \)

KEY WORDS AND PHRASES: Derivation, semiprime ring, prime ring, commutative

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1. INTRODUCTION

Let \( R \) be a ring and \( X \) be a subset of \( R \). Let \( \text{Ann}_r(X) = \{a \in R \mid xa = 0 \text{ all } x \in X\} \) and \( \text{Ann}_l(X) = \{a \in R \mid ax = 0 \text{ all } x \in X\} \) be the right and left annihilators, respectively, of the subset \( X \) of \( R \). If \( R \) is a semiprime ring then the left and right and two-sided annihilators of an ideal \( X \) coincide. It will be denoted by \( \text{Ann}(X) \). Let \( U \) be an ideal of \( R \). Note that if \( \sigma \) is an automorphism of \( R \) and \( \text{Ann}(U) = 0 \) then \( \text{Ann}(\sigma(U)) = 0 \). Let \( R \) be a ring and \( \alpha, \beta \) be two automorphisms of \( R \). An additive mapping \( d : R \to R \) is called an \((\alpha, \beta)\)-derivation if \( d(xy) = \alpha(x)d(y) + d(x)\beta(y) \) holds for all pairs \( x, y \in R \)

Throughout this note \( R \) will represent an associative ring. Let \( R' = \{x \in R \mid d(x) = 0\} \). The centralizer of a subset \( A \) of \( R \) is \( C_R(A) = \{y \in R \mid ay = ya, \forall a \in A\} \). \( C_R(R) = Z \), the center of \( R \)

There are two motivations for this research. Herstein [1] has proved that if \( \sigma \) is an automorphism of \( R \) with \( \text{Ann}(U) = 0 \), then any element \( a \in R \) satisfying \( ad(x) = d(x)a \) for all \( x \in R \), should be central. In [2], Daif has proved the following theorem. Let \( R \) be a prime ring and \( a, b \in R \). Then the following conditions are equivalent

\( i) \ ad(x) = d(x)b \), \( \forall x \in R \)

\( ii) \) Either \( a = b \in C_R(d(R)) \) or \( C_R(a) = C_R(b) = R' \) and \( a[a, x] = [a, x]b \) (or \( a[b, x] = [b, x]b \)) for all \( x \in R \). In the first part of this note we generalized these two theorems for an ideal \( U \) and \((\alpha, \beta)\)-derivation of \( R \).
In the second part, Bresar and Vukman [3] give some results concerning two derivations in semiprime rings. We will generalize some of these results by taking an ideal of $R$ instead of $R$ and extend to more general mappings. As a result of this, we will give a generalization of a well-known result of Posner which states that if $R$ is a prime ring of characteristic not 2 and $d, g$ are nonzero derivation of $R$ then $dg$ cannot be a derivation.

2. RESULTS

**LEMMA 1.** Let $R$ be a prime ring of characteristics not 2, $(0) \neq U$ an ideal of $R$, $0 \neq d : R \rightarrow R$ a $(\alpha, \beta)$-derivation such that $\alpha d = d \alpha, d \beta = \beta d$ and $\alpha \in R$. If $\alpha \in C_R(d(U))$ then $\alpha \in Z$.

**PROOF.** Since $\alpha \in C_R(d(U))$, $d(x) = d(x)a$ for all $x \in U$. Replacing $x$ by $xy$, $y \in U$, we obtain $\alpha \alpha(x)d(y) + d(x)\beta(y) = \alpha(x)d(y)a + d(x)\beta(y)a$. Using hypothesis we have

$$d(x)\alpha(\beta(y)) = \alpha(x), \alpha \beta(y).$$

Taking $yr, r \in R$, instead of $y$, we obtain

$$d(x)\beta(y)[\alpha, \beta(y)] = \alpha(x), \alpha \beta(y)d(r) \text{ for all } x, y \in U, r \in R.$$ If we replace $r$ by $\beta^{-1}(d(z)), z \in U$, we get $d(x)\beta(y)[\alpha, d(z)] = \alpha(x), \alpha \beta(y)\beta^{-1}(d(z)).$ Since $\alpha \in C_R(d(U))$ we have $\alpha(x), \alpha \beta(y)^{-1}(d^2(z)) = 0$ for all $x, y, z \in U$. Since $\alpha(U)$ is an ideal of $R$ and $R$ is prime we get $\alpha \in Z$ or $d^2(U) = 0$. If $d(U) = 0$, then $d^2(xy) = \alpha^2(x)d^2(y) + 2d(x)\beta(y)\beta(y)$ and so $d(x \beta(y)) = 0$. By [4, Lemma 3] we have a contradiction. Thus $\alpha \in Z$.

**THEOREM 1.** Let $R$ be a prime ring of characteristic not 2, $(0) \neq U$ an ideal of $R$ and $a, b \in R$. Then the following conditions are equivalent

1. $a(a)d(x) = d(x)a(a)$, for all $x \in U$.
2. Either $\beta(b) = \alpha(a) \in C_R(d(U))$ or $C_R(a) = C_R(b) = R'$ and $\alpha[a, x] = [a, x]b$ (or $[b, c] = [b, x]b$) for all $x \in U$.

**PROOF.** (I) $\Rightarrow$ (II) If $\alpha \in C_R(d(U))$ then by Lemma 1 we get $\alpha \in Z$. (I) gives $d(x)\beta(b) - \alpha(a)) = 0$, for all $x \in U$. By [4, Lemma 3] it implies that $\beta(b) = \alpha(a)$. Similarly, if $\beta(b) \in C_R(d(U))$ then $\beta(b) = \alpha(a)$.

We assume henceforth that neither $\alpha(a)$ nor $\beta(b)$ in $C_R(d(U))$. Let in (I) $x$ be $rz$, where $r \in R$, and using (I), we have $\alpha(a(\alpha(r))d(x) + \alpha(a)d(r)\beta(x) = \alpha(r)d(x)\beta(b) + d(r)\beta(x)\beta(b)$ and so

$$\alpha([a, r])d(x) = d(r)\alpha(b(x)) - \alpha(a)d(r)\beta(x).$$

(2.1)

Taking $y$ instead of $r$ where $y \in U$, in (2.1) and using (I) we obtain

$$\alpha([a, y])d(x) = d(y)\beta([x, b]), \text{ for all } x, y \in U.$$ (2.2)

Now if $d(x) = 0$ then (2.2) gives us $d(y)\beta([x, b]) = 0$ for all $y \in U$. By [4, Lemma 3], we get $x \in C_R(b)$. Conversely, if $x \in C_R(b)$, then (2.2) gives us $\alpha([y, x])d(x) = 0$. Since by [4, Lemma 3] $a \notin Z$, we have $d(x) = 0$. Therefore $C_R(b) = R'$. Similarly, we can show that $C_R(a) = R'$.

In particular, $d(a) = d(b) = 0$ and $ab = ba$.

Replace $r$ by $yb, y \in U$, in (2.1) we have $\alpha([a, y])\alpha(b)\beta(x) = d(y)\beta(b)(xb) - \alpha(a)d(y)\beta(x) = \alpha(a)d(y)\beta(bx) = \alpha(a)d(y)\beta(x) - \alpha(a)d(y)\beta(x) = \alpha(a)d(y)\beta([x, b])$ and using (2.2) we get $\alpha([a, y])\alpha(b)d(x) = \alpha(a)\alpha([a, y])d(x)$ and so

$$\alpha([a, y])b - \alpha([a, y])d(x) = 0 \text{ for all } x, y \in U.$$ By [4, Lemma 3] we obtain

$$a[a, y] = [a, y]b \text{ for all } y \in U.$$
Furthermore, replacing $x$ by $ax$ in (2.2) and using (2.2) and hypothesis we also have $a[b, x] = [b, x]b$

(II) $\Rightarrow$ (I) If $\alpha(a) = \beta(b) \in C_R(d(U))$ it is obviously $\alpha(a)d(x) = d(x)\beta(b)$ for all $x \in U$. Therefore it suffices to show that if $C_R(a) = C_R(b) = R'$ and $a[a, x] = [a, x]b$ for all $x \in U$ then $\alpha(a)d(x) = d(x)\beta(b)$ for all $x \in U$.

Since $d(a) = d(b) = 0$, $ab = ba$, $[a, ax - xb] = a[a, x] - [a, x]b = 0$ It gives $ax - xb \in R'$ and so $0 = d(ax - xb) = \alpha(a)d(x) - d(x)\beta(b)$. This proves the theorem.

For the second part we begin with

**Lemma 2** [3, Lemma 1]. Let $R$ be a 2-torsion free semiprime ring and $a, b$ the elements of $R$. Then the following conditions are equivalent:

(i) $axb = 0$ for all $x \in R$
(ii) $bxa = 0$ for all $x \in R$
(iii) $axb + bxa = 0$ for all $x \in R$

If one of these conditions is fulfilled then $ab = ba = 0$ too.

**Lemma 3.** Let $R$ be a semiprime ring and $U$ a nonzero ideal of $R$ such that $Ann(U) = 0$. Let $d$ be an $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$. If $d(U)Ug(U) = 0$ then $d(R)Ug(R) = 0$.

**Proof.** For all $x, y, z \in U$, $d(x)yg(z) = 0$. Replace $x$ by $xs$, $s \in R$ we have $0 = d(xs)yg(z) = \alpha(x)d(s)yg(z) + d(x)\beta(s)yg(z)$. Since $\beta(s)y \in U$, the last equation implies that $\alpha(x)d(s)yg(z) = 0$, for all $x, y, z \in U$ and $s \in R$. Taking $tz$ instead of $z$, where $t \in R$, we have $0 = \alpha(x)d(s)yg(t)\beta(z) + d(x)\beta(s)yg(t)\delta(z)$. Since $\gamma(t) \in U$, it gives $\alpha(x)d(s)yg(t)\delta(z) = 0$ for all $x, y, z \in U$ and $s, t \in R$. Therefore $d(s)yg(t)\delta(z) \in Ann(\alpha(U)) = 0$. Thus we get $d(s)yg(t)\delta(z) = d(s)yg(t)\delta(z)\gamma(t)g(z) = 0$. As a result of this, it implies that $d(R)Ug(R) = 0$.

**Lemma 4.** Let $R$ be a semiprime ring and $U$ be a nonzero ideal of $R$ such that $Ann(U) = 0$. Let $a, b \in R$ be such that $aUb = 0$ then $aRb = 0$.

**Proof.** For all $x \in U$, $0 = axb$. Replace $x$ by $tbxrat$, where $t, r \in r$ we have $atb = atb = 0$. Since $R$ is semiprime ring, this implies that $atbU = 0$ for all $t \in R$. Thus $atb \in Ann(U) = 0$ we get $aRb = 0$.

**Theorem 2.** Let $R$ be a 2-torsion free semiprime ring and $U$ be a nonzero ideal of $R$ with $Ann(U) = 0$. Let $d$ be a $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$. Suppose that $dg$ is a $(\alpha_\gamma, \beta_\delta)$-derivation and $g$ commutes both $\gamma$ and $\delta$. Then $g(x)Ug^{-1}(y) = 0$, for all $x, y \in U$.

**Proof.** Since $g$ commutes both $\gamma$ and $\delta$, from the first par to the proof of [5, Lemma 1] there is no loss of generality in assuming $\beta = 1$ and $\delta = 1$. For all $x, y, z \in U$, $dg(xy) = d(\gamma(x)g(y) + g(x)y) = \alpha_\gamma(x)dg(y) + d(\gamma(x))g(y) + \alpha(g(x))d(y) + dg(x)y$. On the other hand, since $dg$ is an $(\alpha_\gamma, 1)$-derivation we have $dg(xy) = \gamma(x)dg(y) + dg(x)y$. Comparing the two expressions so obtained for $dg(xy)$, we see that

$$d(\gamma(x))g(y) + \alpha(g(x))d(y) = 0 \quad \text{for all} \quad x, y \in U. \quad (2.3)$$

Replacing $y$ by $yz$ where $z \in R$ in (2.3) we obtain $0 = d(\gamma(x))g(yz) + \alpha(g(x))d(yz) = d(\gamma(x))\gamma(y)g(z) + d(\gamma(x))g(y)z + \alpha(g(x))\alpha(y)d(z) + \alpha(g(x))d(y)z = [d(\gamma(x))g(y) + \alpha(g(x))d(y)]z + d(\gamma(x))\gamma(y)g(z) + \alpha(g(x))\alpha(y)d(z)$. This relation reduces to

$$d(\gamma(x))\gamma(y)g(z) + \alpha(g(x))\alpha(y)d(z) = 0 \quad \text{for all} \quad x, y, z \in U, z \in R. \quad (2.4)$$

Replace $y$ by $yg(t)$, $t \in U$ and take $z \in U$ we have $d(\gamma(x))\gamma(y)g(t)g(z) + \alpha(g(x))\alpha(y)d(g(t))d(z) = 0$. Considering this relation (2.4) and (2.3) we obtain $d(\gamma(x))\gamma(y)g(t)g(z) = -\alpha(g(x))\alpha(y)d(g(t))g(z) = \alpha(g(x))\alpha(y)d(g(t))d(z)$ for all $x, y, z \in U$. Comparing the last two relations we get

$$2\alpha(g(x))\alpha(y)\alpha(g(t))d(z) = 0. \quad \text{Since} \quad R \quad \text{is} \quad 2\text{-torsion free, it gives}$$
Replacing \( t \) by \( tu, u \in U \) it follows
\[
g(x)yg(t)\alpha^{-1}d(z) = 0 \quad \text{for all } x, y, z, t \in U.
\]
Since \( g(t) \in U \) this relation reduces to
\[
g(x)\alpha^{-1}(d(z)) = 0 \quad \text{for all } x, t, u, z \in U.
\]
By Lemma 4 we have for all \( x, t, u, z \in U, g(x)Rg(t)\alpha^{-1}(d(z)) = 0 \).
In particular \( g(x)\alpha^{-1}(d(z))Rg(x)\alpha^{-1}(d(z)) = 0 \) for all \( x, u, z \in U \).
Since \( R \) is semiprime we obtain \( g(x)\alpha^{-1}(d(z)) = 0 \) for all \( z, z \in U \).

**Corollary.** Let \( R \) be a prime ring of characteristic not 2, \( d \) be an \( (\alpha, \beta) \)-derivation of \( R \) and \( g \) be a \( (\gamma, \delta) \)-derivation of \( R \) such that \( g \) commutes both \( \gamma \) and \( \delta \). If the composition \( dg \) is a \( (\alpha\gamma, \beta\delta) \)-derivation then \( d = 0 \) or \( g = 0 \).

**Theorem 3.** Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a nonzero ideal of \( R \) such that \( \text{Ann}(U) = 0 \). Let \( d \) be a \( (\alpha, \beta) \)-derivation of \( R \) and \( g \) be a \( (\gamma, \delta) \)-derivation of \( R \) such that \( g \) commutes both \( \gamma \) and \( \delta \). If for all \( x, y \in U, \) \( \beta^{-1}(d(x))Ug(y) = 0 = g(x)\alpha^{-1}(d(y)) \) then \( dg \) is a \( (\alpha\gamma, \beta\delta) \)-derivation on \( R \).

**Proof.** If \( x, y, z \in R \) \( \beta^{-1}(d(x))yg(z) = 0 \) \( g(x)\alpha^{-1}(d(z)) = 0 \) for all \( x, y, z \in R \) and since \( g \) commutes both \( \gamma \) and \( \delta \), \( \beta^{-1}(d(x))yg(z) = 0 \) for all \( x, y, z \in R \). Since \( R \) is a semiprime ring, by Lemma 2 we obtain \( d(\gamma(x))\beta(g(z)) = 0 \) for all \( x, z \in R \). Similarly from \( g(x)\alpha^{-1}(d(y)) = 0 \), we get \( \alpha(g(x))d(\delta(y)) = 0 \) Therefore \( dg \) is an \( (\alpha\gamma, \beta\delta) \)-derivation on \( R \).

**References**


