

ASYMPTOTIC BEHAVIOR OF ALMOST-ORBITS OF REVERSIBLE SEMIGROUPS OF NON-LIPSCHITZIAN MAPPINGS IN BANACH SPACES

JONG SOO JUNG

Department of Mathematics
Dong-A University
Pusan 607-714, KOREA
E-mail address: jungjs@seunghak.donga.ac.kr

JONG YEOUL PARK

Department of Mathematics
Pusan National University
Pusan 609-735, KOREA

JONG SEO PARK

Department of Mathematics
Graduate School, Dong-A University
Pusan 607-714, KOREA

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ABSTRACT. Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, G a right reversible semitopological semigroup, and $S = \{S(t) : t \in G\}$ a continuous representation of G as mappings of asymptotically nonexpansive type of C into itself. The weak convergence of an almost-orbit $\{u(t) : t \in G\}$ of $S = \{S(t) : t \in G\}$ on C is established. Furthermore, it is shown that if P is the metric projection of E onto set $F(S)$ of all common fixed points of $S = \{S(t) : t \in G\}$, then the strong limit of the net $\{Pu(t) : t \in G\}$ exists.

KEY WORDS AND PHRASES: Almost-orbit, fixed point, reversible semitopological semigroup, semigroup of asymptotically nonexpansive type, uniformly convex Banach space

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1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Banach space E and let $S = \{S(t) : t \geq 0\}$ be a family of mappings from C into itself such that $S(0) = I$, $S(t+s) = S(t)S(s)$ for all $t, s \in [0, \infty)$ and $S(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$. S is said to be

- (a) nonexpansive semigroup on C if $\|S(t)x - S(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \geq 0$,
- (b) asymptotically nonexpansive semigroup on C [1] if there is a function $k : [0, \infty) \rightarrow [0, \infty)$ with $\limsup_{t \rightarrow \infty} k(t) \leq 1$ such that $\|S(t)x - S(t)y\| \leq k(t)\|x - y\|$ for all $x, y \in C$ and $t \geq 0$,
- (c) semigroup of asymptotically nonexpansive type on C [1] if for each $x \in C$,

$$\limsup_{t \rightarrow \infty} \left\{ \sup_{y \in C} [\|S(t)x - S(t)y\| - \|x - y\|] \right\} \leq 0;$$

see [2] for mappings of asymptotically nonexpansive type. It is easily seen that (a) \Rightarrow (b) \Rightarrow (c) and that both the inclusions are proper (cf. [1, p. 112]).

In [3], Myadera and Kobayashi introduced the notion of almost-orbits of nonexpansive semigroups on C and provided the weak and strong almost convergences of such an almost-orbit in a uniformly convex Banach space; see also [4] for almost-orbits of nonexpansive mappings. Recently, Tan and Xu [5] extended this notion to semigroups of asymptotic nonexpansive type in Hilbert spaces. The case of

general commutative nonexpansive semigroups in uniformly convex Banach spaces was studied by Takahashi and Park [11]. Oka [6] gave the results for the case of commutative asymptotically nonexpansive semigroups in uniformly convex Banach spaces. In particular, Takahashi and Zhang [7] established the convergences of almost-orbits of noncommutative asymptotically nonexpansive semigroups in the same Banach spaces, see [8] for the case of Hilbert spaces

The purpose of this paper is to generalize their results to the case of noncommutative semigroups of asymptotically nonexpansive type. Section 2 is a preliminary part. In Section 3, we prove several lemmas which are crucial for our discussion. Main results are given in Section 4. First, we establish the weak convergence (Theorem 1) of an almost-orbit $\{u(t) : t \in G\}$ of a semigroup $S = \{S(t) : t \in G\}$ of asymptotically nonexpansive type on C in a uniformly convex Banach space with a Fréchet differentiable norm, where G is a right reversible semitopological semigroup. Next, we show that if P is the metric projection of E onto set $F(S)$ of all common fixed points of $S = \{S(t) : t \in G\}$, then the strong limit of the net $\{Pu(t) : t \in G\}$ exists (Theorem 2). Our proofs employ the methods of Hirano and Takahashi [9], Ishihara and Takahashi [10], Takahashi and park [11], and Takahashi and Zhang [7,8]. The results are generalizations of the corresponding results in [5], [7], [8], [11], [12] and [13].

2. PRELIMINARIES

Let E be a real Banach space and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by (x, f) . With each $x \in E$, we associate the set

$$J(x) = \{f \in E^* : (x, f) = \|x\|^2 = \|f\|^2\}.$$

Using the Hahn-Banach theorem, it is readily verified that $J(x) \neq \emptyset$. The multivalued mapping $J : E \rightarrow E^*$ is called the duality mapping of E . Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . Then a Banach space E is said to be smooth provided the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for each $x, y \in U$. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. It is also known that if E is smooth, then the duality mapping J is single valued. It is easy to see that the norm of E is Fréchet differentiable if and only if for any bounded set $B \subset E$ and any $x \in E$, $\lim_{t \rightarrow 0} (2t)^{-1} (\|x + ty\|^2 - \|x\|^2) = (y, J(x))$ uniformly in $y \in B$; see [14].

A Banach space E is called uniformly convex if the modulus of convexity

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

is positive in its domain of definition $\{\epsilon : 0 < \epsilon \leq 2\}$. For the properties of $\delta(\epsilon)$, see [15].

For a subset D of E , \overline{D} denotes the closure of D , coD the convex hull of D , and \overline{coD} the closed convex hull of D , respectively.

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $a \in G$ the mappings $g \rightarrow a \cdot g$ and $g \rightarrow g \cdot a$ from G to G are continuous. G is said to be right reversible if any two closed left ideals of G have nonempty intersection. If G is right reversible, (G, \preceq) is a directed system when the binary relation " \preceq " on G is defined by $a \preceq b$ if and only if $\{a\} \cup \overline{Ga} \supseteq \{b\} \cup \overline{Gb}$.

Let C be a nonempty closed convex subset of a Banach space E and let G be a semitopological semigroup. A family $S = \{S(t) : t \in G\}$ of mappings from C into itself is said to be a (continuous) representation of G on C if S satisfies the following:

- (i) $S(ts)x = S(t)S(s)x$ for all $t, s \in G$ and $x \in C$
- (ii) for every $x \in C$, the mappings $s \rightarrow S(s)x$ from G into C is continuous.

DEFINITION 1. A representation $\mathcal{S} = \{S(t) : t \in G\}$ of G on C is said to be a semigroup of asymptotically nonexpansive type on C if for each $x \in C$,

$$\inf_{s \in G} \sup_{s \preceq t} \sup_{y \in C} (\|S(t)x - S(t)y\| - \|x - y\|) \leq 0. \quad (2.2)$$

Let G be right reversible and let $\mathcal{S} = \{S(t) : t \in G\}$ be a representation of G on C . A function $u : G \rightarrow C$ is called an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$ if

$$\lim_{s \in G} \left(\sup_{t \in G} \|u(ts) - S(t)u(s)\| \right) = 0. \quad (2.3)$$

$\omega(u)$ denotes the set of all weak limit points of subnets of the net $\{u(t) : t \in G\}$, and $F(\mathcal{S}) = \bigcap_{t \in G} F(S(t))$ the set of all common fixed points of mappings $S(t)$, $t \in G$ in C .

3. LEMMAS

In this section, we prove several lemmas which are crucial in convergence of almost-orbits.

LEMMA 1. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type of a right reversible semitopological semigroup G on C . Then $F(\mathcal{S})$ is a closed and convex subset of C .

PROOF. The closedness of $F(\mathcal{S})$ is obvious. To show convexity, it is sufficient to show that $z = \frac{x+y}{2} \in F(\mathcal{S})$ for all $x, y \in F(\mathcal{S})$. Let $x, y \in F(\mathcal{S})$, $x \neq y$. If $\lim_{t \in G} S(t)z = z$, then for any $s \in G$,

$$S(s)z = \lim_{t \in G} S(s)S(t)z = \lim_{t \in G} S(st)z = \lim_{t \in G} S(t)z = z,$$

i.e., $z \in F(\mathcal{S})$. Hence it suffices to prove that $\lim_{t \in G} S(t)z = z$. If not, there exists $\epsilon > 0$ such that for any $t \in G$, there is $t' \in G$ with $t' \succeq t$ and

$$4\|S(t')z - z\| = \|2(S(t')z - x) - 2(y - S(t')z)\| \geq \epsilon.$$

Choose $d > 0$ so small that

$$(R + d) \left(1 - \delta \left(\frac{\epsilon}{R + d} \right) \right) < R,$$

where $R = \|x - y\| > 0$ and δ is the modulus of convexity of E . Since $\mathcal{S} = \{S(t) : t \in G\}$ is asymptotically nonexpansive type on C , there is $t_0 \in G$ such that

$$\sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)z - S(t)w\| - \|z - w\|) \leq \frac{d}{2}.$$

Put $u = 2(S(t'_0)z - x)$, $v = 2(y - S(t'_0)z)$. Then $\|u - v\| = 4\|S(t'_0)z - z\| \geq \epsilon$. Furthermore, since $t_0 \preceq t'_0$, we have

$$\begin{aligned} \|u\| &= 2\|S(t'_0)z - x\| \\ &= 2(\|S(t'_0)z - S(t'_0)x\| - \|z - x\|) + 2\|z - x\| \\ &\leq 2 \sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)z - S(t)w\| - \|z - w\|) + \|x - y\| < R + d \end{aligned}$$

and

$$\begin{aligned} \|v\| &= 2\|y - S(t'_0)z\| \\ &= 2(\|S(t'_0)z - S(t'_0)y\| - \|z - y\|) + 2\|z - y\| \\ &\leq 2 \sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)z - S(t)w\| - \|z - w\|) + \|x - y\| < R + d. \end{aligned}$$

So we have

$$\left\| \frac{u + v}{2} \right\| \leq (R + d) \left(1 - \delta \left(\frac{\epsilon}{R + d} \right) \right),$$

and hence

$$\|x - y\| = \left\| \frac{u + v}{2} \right\| \leq (R + d) \left(1 - \delta \left(\frac{\epsilon}{R-d} \right) \right) < R = \|x - y\|.$$

This is a contraction. Therefore, $\lim_{t \in G} S(t)z = z$, which completes the proof

LEMMA 2. Let C be a nonempty closed convex subset of Banach space E . Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C . If $\{u(t) : t \in G\}$ and $\{v(t) : t \in G\}$ are almost-orbits of $\mathcal{S} = \{S(t) : t \in G\}$, then $\lim_{t \in G} \|u(t) - v(t)\|$ exists. In particular, for every $z \in F(\mathcal{S})$, $\lim_{t \in G} \|u(t) - z\|$ exists

PROOF. Put

$$\phi(s) = \sup_{t \in G} \|u(ts) - S(t)u(s)\|, \quad \psi(s) = \sup_{t \in G} \|v(ts) - S(t)v(s)\|$$

for $s \in G$. Then $\lim_{s \in G} \phi(s) = \lim_{s \in G} \psi(s) = 0$. Let $\epsilon > 0$. Since $\mathcal{S} = \{S(t) : t \in G\}$ is of asymptotically nonexpansive type on C , there exists $t_0 \in G$ such that

$$\sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) < \epsilon$$

for all $s \in G$. On the other hand, since, for any $s, t \in G$,

$$\begin{aligned} \|u(ts) - v(ts)\| &\leq \phi(s) + \psi(s) + (\|S(t)u(s) - S(t)v(s)\| - \|u(s) - v(s)\|) + \|u(s) - v(s)\| \\ &\leq \phi(s) + \psi(s) + \sup_{w \in C} (\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) + \|u(s) - v(s)\|, \end{aligned}$$

we have

$$\begin{aligned} \inf_{t \in G} \sup_{t \preceq \tau} \|u(\tau) - v(\tau)\| &\leq \phi(s) + \psi(s) + \sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) + \|u(s) - v(s)\| \\ &\leq \phi(s) + \psi(s) + \epsilon + \|u(s) - v(s)\|, \end{aligned}$$

and then $\inf_{t \in G} \sup_{t \preceq \tau} \|u(\tau) - v(\tau)\| \leq \sup_{t \in G} \inf_{t \preceq s} \|u(s) - v(s)\|$. Thus $\lim_{t \in G} \|u(t) - v(t)\|$ exists. Let $z \in F(\mathcal{S})$ and put $v(t) = z$. Then $v(t)$ is an almost-orbit and hence $\lim_{t \in G} \|u(t) - z\|$ exists

LEMMA 3. Let C be a nonempty closed convex subset of Banach space E . Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C . Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. If $F(\mathcal{S}) \neq \emptyset$, then there exists $t_0 \in G$ such that $\{u(t) : t \succeq t_0\}$ is bounded.

PROOF. Let $z \in F(\mathcal{S})$. Then, since $\lim_{t \in G} \|u(t) - z\|$ exists by Lemma 2, there is $t_0 \in G$ such that $\{\|u(t) - z\| : t \succeq t_0\}$ is bounded. Hence $\{u(t) : t \succeq t_0\}$ is bounded.

LEMMA 4. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C . Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Suppose that $F(\mathcal{S}) \neq \emptyset$. Let $y \in F(\mathcal{S})$ and $0 < \alpha \leq \beta < 1$. Then for any $\epsilon > 0$, there is $t_0 \in G$ such that

$$\|S(t)(\lambda u(s) + (1 - \lambda)y) - (\lambda S(t)u(s) + (1 - \lambda)y)\| < \epsilon$$

for all $t, s \succeq t_0$ and $\lambda \in [\alpha, \beta]$.

PROOF. By Lemma 2, $\lim_{t \in G} \|u(t) - y\|$ exists. Let $\epsilon > 0$ and

$$r = \lim_{t \in G} \|u(t) - y\|.$$

If $r = 0$, since $\mathcal{S} = \{S(t) : t \in G\}$ is of asymptotically nonexpansive type on C , there exists $t_0 \in G$ such that

$$\sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)(\lambda u(s) + (1 - \lambda)y) - S(t)w\| - \|\lambda u(s) + (1 - \lambda)y - w\|) < \frac{\epsilon}{2},$$

and

$$\|u(t) - y\| < \frac{\epsilon}{4}$$

for $t \geq t_0$, $0 < \lambda < 1$ and $s \in G$. Hence for $s, t \geq t_0$, $0 < \lambda < 1$ and $s \in G$,

$$\begin{aligned}
& \|S(t)(\lambda u(s) + (1-\lambda)y) - (\lambda S(t)u(s) + (1-\lambda)y)\| \\
& \leq \lambda \|S(t)(\lambda u(s) + (1-\lambda)y) - S(t)u(s)\| + (1-\lambda) \|S(t)(\lambda u(s) + (1-\lambda)y) - y\| \\
& \leq \lambda \left(\sup_{t_0 \leq t} \sup_{w \in C} (\|S(t)(\lambda u(s) + (1-\lambda)y) - S(t)w\| - \|\lambda u(s) + (1-\lambda)y - w\|) \right) \\
& \quad + \lambda \|\lambda u(s) + (1-\lambda)y - u(s)\| + (1-\lambda) \left(\sup_{t_0 \leq t} \sup_{w \in C} (\|S(t)(\lambda u(s) + (1-\lambda)y) - \right. \\
& \quad \left. S(t)w\| - \|\lambda u(s) + (1-\lambda)y - w\|) \right) + (1-\lambda) \|\lambda u(s) + (1-\lambda)y - y\| \\
& < \lambda \frac{\epsilon}{2} + (1-\lambda) \frac{\epsilon}{2} + 2\lambda(1-\lambda) \|u(s) - y\| \\
& < \frac{\epsilon}{2} + \frac{\epsilon}{2} (\lambda(1-\lambda)) < \epsilon.
\end{aligned}$$

Now, let $r > 0$. Then we can choose $d > 0$ so small that

$$(r+d) \left(1 - c\delta \left(\frac{\epsilon}{r+d} \right) \right) = r_0 < r,$$

where δ is the modulus of convexity of E and

$$c = \min\{2\lambda(1-\lambda) : \alpha \leq \lambda \leq \beta\}.$$

Let $a > 0$ with $2a + r_0 < r$. Then there is $t_0 \in G$ such that

$$r - a < \|u(s) - y\| \leq r + \frac{d}{2} \quad \text{for } s \geq t_0,$$

$$\|S(s)u(t) - u(st)\| < a \quad \text{for } t \geq t_0 \quad \text{and } s \in G,$$

$$\sup_{t_0 \leq t} \sup_{w \in C} (\|S(t)z - S(t)w\| - \|z - w\|) < \frac{c}{4} d \quad \text{for } z \in C,$$

and

$$\sup_{t_0 \leq t} \sup_{w \in C} (\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) < \frac{c}{4} d \quad \text{for } s \in G.$$

Suppose that $\|S(t)(\lambda u(s) + (1-\lambda)y) - (\lambda S(t)u(s) + (1-\lambda)y)\| \geq \epsilon$ for some $s, t \geq t_0$ and $\lambda \in [\alpha, \beta]$. Put $z = \lambda u(s) + (1-\lambda)y$, $u = (1-\lambda)(S(t)z - y)$ and $v = \lambda(S(t)u(s) - S(t)z)$. Then we have

$$\begin{aligned}
\|u\| &= (1-\lambda) (\|S(t)z - S(t)y\| - \|z - y\|) + (1-\lambda) \|z - y\| \\
&\leq (1-\lambda) \sup_{t_0 \leq t} \sup_{w \in C} (\|S(t)z - S(t)w\| - \|z - w\|) + (1-\lambda) \|\lambda u(s) + (1-\lambda)y - y\| \\
&< (1-\lambda) \frac{c}{4} d + \lambda(1-\lambda) \|u(s) - y\| \\
&\leq \lambda(1-\lambda) \left((1-\lambda) \frac{d}{2} + r + \frac{d}{2} \right) < \lambda(1-\lambda)(r+d)
\end{aligned}$$

and

$$\begin{aligned}
\|v\| &= \lambda (\|S(t)u(s) - S(t)z\| - \|u(s) - z\|) + \lambda \|u(s) - z\| \\
&\leq \lambda \sup_{t_0 \leq t} \sup_{w \in C} (\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) + \lambda(1-\lambda) \|u(s) - y\| \\
&< \lambda \frac{c}{4} d + \lambda(1-\lambda) \left(r + \frac{d}{2} \right) \\
&\leq \lambda(1-\lambda) \left(\lambda \frac{d}{2} + r + \frac{d}{2} \right) < \lambda(1-\lambda)(r+d).
\end{aligned}$$

We also have that

$$\|u - v\| = \|S(t)z - (\lambda S(t)u(s) + (1-\lambda)y)\| \geq \epsilon$$

and

$$\begin{aligned}\lambda u + (1 - \lambda)v &= \lambda(1 - \lambda)(S(t)z - y) + (1 - \lambda)\lambda(S(t)u(s) - S(t)z) \\ &= \lambda(1 - \lambda)(S(t)u(s) - y).\end{aligned}$$

By the Lemma in [16], we have

$$\begin{aligned}\lambda(1 - \lambda)\|S(t)u(s) - y\| &= \|\lambda u + (1 - \lambda)v\| \\ &\leq \lambda(1 - \lambda)(r + d)\left(1 - 2\lambda(1 - \lambda)\delta\left(\frac{\epsilon}{r + d}\right)\right) \\ &\leq \lambda(1 - \lambda)(r + d)\left(1 - c\delta\left(\frac{\epsilon}{r + d}\right)\right) = \lambda(1 - \lambda)r_0\end{aligned}$$

and hence $\|S(t)u(s) - y\| \leq r_0$. This implies that

$$\|u(ts) - y\| \leq \|u(ts) - S(t)u(s)\| + \|S(t)u(s) - y\| < a + r_0 < r - a.$$

This contradicts the fact $\|u(s) - y\| > r - a$ for $s \geq t_0$. The proof is complete

LEMMA 5. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C . Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Suppose that $F(\mathcal{S}) \neq \emptyset$. Then $\lim_{t \in G} \|\lambda u(t) + (1 - \lambda)x - y\|$ exists for every $x, y \in F(\mathcal{S})$.

PROOF. Let $\lambda \in (0, 1)$ and $x, y \in F(\mathcal{S})$. By (2.2), (2.3), and Lemma 4, for any $\epsilon > 0$, there exists $t_0 \in G$ such that

$$\begin{aligned}\|S(t)(\lambda u(s) + (1 - \lambda)x) - (\lambda S(t)u(s) + (1 - \lambda)x)\| &\leq \frac{\epsilon}{3} \quad \text{for } t, s \geq t_0, \\ \sup_{t \in G} \|u(ts) - S(t)u(s)\| &< \frac{\epsilon}{3} \quad \text{for } s \geq t_0, \\ \sup_{t_0 \leq t} \sup_{w \in C} (\|S(t)(\lambda u(s) + (1 - \lambda)x) - S(t)w\| - \|\lambda u(s) + (1 - \lambda)x - w\|) &< \frac{\epsilon}{3} \quad \text{for } s \in G.\end{aligned}$$

Since

$$\begin{aligned}\|\lambda u(ts) + (1 - \lambda)x - y\| &\leq \lambda\|u(ts) - S(t)u(s)\| + \|\lambda S(t)u(s) + (1 - \lambda)x - S(t)(\lambda u(s) + (1 - \lambda)x)\| \\ &\quad + \sup_{w \in C} (\|S(t)(\lambda u(s) + (1 - \lambda)x) - S(t)w\| - \|\lambda u(s) + (1 - \lambda)x - w\|) \\ &\quad + \|\lambda u(s) + (1 - \lambda)x - y\| \\ &< \epsilon + \|\lambda u(s) + (1 - \lambda)x - y\|\end{aligned}$$

for all $t, s \in G$, we have

$$\begin{aligned}\inf_{t \in G} \sup_{t \leq \tau} \|\lambda u(\tau) + (1 - \lambda)x - y\| &\leq \sup_{t_0 \leq t} \|\lambda u(ts) + (1 - \lambda)x - y\| \\ &\leq \epsilon + \|\lambda u(s) + (1 - \lambda)x - y\|\end{aligned}$$

for all $s \geq t_0$, and then

$$\inf_{t \in G} \sup_{t \leq \tau} \|\lambda u(\tau) + (1 - \lambda)x - y\| \leq \sup_{t \in G} \inf_{t \leq s} \|\lambda u(s) + (1 - \lambda)x - y\|.$$

Thus $\lim_{t \in G} \|\lambda u(t) + (1 - \lambda)x - y\|$ exists.

LEMMA 6. Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C . Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Then

$$F(\mathcal{S}) \cap \bigcap_{s \in G} \overline{\text{co}}\{u(t) : t \geq s\}$$

is at most a singleton.

PROOF. Note that $\bigcap_{s \in G} \overline{\omega} \{u(t) : t \succeq s\} = \overline{\omega} \omega(u)$, see [17]. Let $x, y \in F(\mathcal{S})$. Since E has a Fréchet differentiable norm, there exists an increasing function $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\gamma(t)/t \rightarrow 0$ as $t \rightarrow 0^+$, and

$$\begin{aligned} \frac{1}{2} \|x - y\|^2 + (h, J(x - y)) &\leq \frac{1}{2} \|x - y + h\|^2 \\ &\leq \frac{1}{2} \|x - y\|^2 + (h, J(x - y)) + \gamma(\|h\|) \end{aligned}$$

for all $h \in E$. Take $h = \lambda(u(t) - x)$. Then

$$\begin{aligned} \frac{1}{2} \|x - y\|^2 + \lambda(u(t) - x, J(x - y)) &\leq \frac{1}{2} \|\lambda u(t) + (1 - \lambda)x - y\|^2 \\ &\leq \frac{1}{2} \|x - y\|^2 + \lambda(u(t) - x, J(x - y)) + \gamma(\lambda \|u(t) - x\|). \end{aligned}$$

Using Lemma 5, we have

$$\begin{aligned} \frac{1}{2} \|x - y\|^2 + \lambda \inf_{t \in G} \sup_{t \preceq \tau} (u(\tau) - x, J(x - y)) \\ \leq \frac{1}{2} \lim_{t \in G} \|\lambda u(t) + (1 - \lambda)x - y\|^2 \\ \leq \frac{1}{2} \|x - y\|^2 + \lambda \sup_{t \in G} \inf_{t \preceq \tau} (u(\tau) - x, J(x - y)) + \gamma(\lambda M), \end{aligned}$$

where $\sup_{t \in G} \|u(t) - x\| = M$. Dividing by λ and letting $\lambda \rightarrow 0^+$, we have $\lim_{t \in G} (u(t), J(x - y)) = r$ exists. Of course $r = (v, J(x - y))$ for all $v \in \omega(u)$ and hence for all $v \in \overline{\omega} \omega(u)$. Therefore $(v - w, J(x - y)) = 0$ for all $v, w \in \overline{\omega} \omega(u)$, and it readily follows that $F(\mathcal{S}) \cap \bigcap_{s \in G} \{u(t) : t \succeq s\} = F(\mathcal{S}) \cap \overline{\omega} \omega(u)$ is at most a singleton.

4. MAIN RESULTS

In this section, we study the convergence of an almost-orbit $\{u(t) : t \in G\}$ of $\mathcal{S} = \{S(t) : t \in G\}$

THEOREM 1. Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E . Let F be a subset of C and let G be a right reversible semitopological semigroup. Let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C and let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Assume that

(a) $F \subset F(\mathcal{S})$.

Assume also that

(b) if a subnet $\{u(t_\alpha)\}$ of the net $\{u(t) : t \in G\}$ converges weakly to z , then $z \in F$.

Then either (i) $F = \emptyset$ and $\|u(t)\| \rightarrow \infty$ or (ii) $F \neq \emptyset$ and the net $\{u(t) : t \in G\}$ converges weakly to some $z \in F(\mathcal{S})$.

PROOF. Suppose that some subnet $\{u(t_\alpha)\}$ of $\{u(t) : t \in G\}$ is bounded. Since E is reflexive, a subnet of $\{u(t_\alpha)\}$ must converge weakly to an element $z \in E$, which is in F by (b). Thus $F = \emptyset$ implies $\|u(t)\| = \infty$.

If, on the other hand, $F \neq \emptyset$, then by Lemma 3, $\{u(t) : t \in G\}$ is bounded. So $\{u(t) : t \in G\}$ must contain a subnet $\{u(t_\alpha)\}$ which converges to some $z \in F$ by (b). Since $F \subset F(\mathcal{S})$ and $z \in \overline{\omega} \omega(u) = \bigcap_{s \in G} \overline{\omega} \{u(t) : t \in G\}$, we have

$$z \in F \cap \bigcap_{s \in G} \overline{\omega} \{u(t) : t \succeq s\} \subset F(\mathcal{S}) \cap \bigcap_{s \in G} \overline{\omega} \{u(t) : t \succeq s\}.$$

Therefore it follows from Lemma 6 that $\{u(t) : t \in G\}$ converges weakly to $z \in F(\mathcal{S})$.

As a direct consequence, we have the following corollary, which is a generalization of a result in [5], [7], [8], [11], [12] and [13]

COROLLARY 1. Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E . Let G be a right reversible semitopological

semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C . Suppose that $F(\mathcal{S}) \neq \emptyset$ and let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. If $\omega(u) \subset F(\mathcal{S})$, then the net $\{u(t) : t \in G\}$ converges weakly to some $z \in F(\mathcal{S})$.

PROOF. The result follows by putting $F = \omega(u)$ in Theorem 1.

The following theorem is also a generalization of [7, Theorem 4].

THEOREM 2. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C . Suppose that $F(\mathcal{S}) \neq \emptyset$ and let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Let P denote the metric projection of E onto $F(\mathcal{S})$. Then the strong limit of the net $\{Pu(t) : t \in G\}$ exists and $\lim_{t \in G} Pu(t) = z_0$, where z_0 is a unique element of $F(\mathcal{S})$ such that

$$\lim_{t \in G} \|u(t) - z_0\| = \min \left\{ \lim_{t \in G} \|u(t) - z\| : z \in F(\mathcal{S}) \right\}.$$

PROOF. Since $F(\mathcal{S}) \neq \emptyset$, we know that $\{u(t) : t \in G\}$ is bounded and $\lim_{t \in G} \|u(t) - z\| = g(z)$ exists for each $z \in F(\mathcal{S})$. Let $R = \inf\{g(z) : z \in F(\mathcal{S})\}$ and $M = \{u \in F(\mathcal{S}) : g(u) = R\}$. Then, since $g(z)$ is convex and continuous on $F(\mathcal{S})$ and $g(z) \rightarrow \infty$ as $\|z\| \rightarrow \infty$, M is a nonempty closed convex bounded subset of $F(\mathcal{S})$. Fix $z_0 \in M$ with $g(z_0) = R$. Since P is the metric projection of E onto $F(\mathcal{S})$, we have $\|u(t) - Pu(t)\| \leq \|u(t) - y\|$ for all $t \in G$ and $y \in F(\mathcal{S})$, and hence

$$\inf_{t \in G} \sup_{t \leq s} \|u(s) - Pu(s)\| \leq R.$$

Suppose that $\inf_{t \in G} \sup_{t \leq s} \|u(s) - Pu(s)\| < R$. Then we may choose $\epsilon > 0$ and $t_0 \in G$ such that

$$\|u(s) - Pu(s)\| \leq R - \epsilon$$

$$\sup_{t_0 \leq t} \sup_{w \in C} (\|S(t)u(s) - S(t)w\| - \|u(t) - w\|) < \frac{\epsilon}{4},$$

and

$$\sup_{t \in G} \|u(ts) - S(t)u(s)\| < \frac{\epsilon}{4}$$

for all $s \geq t_0$. Since

$$\begin{aligned} \|u(ts) - Pu(s)\| &\leq \|u(ts) - S(t)u(s)\| + \|S(t)u(s) - S(t)Pu(s)\| \\ &\quad - \|u(s) - Pu(s)\| + \|u(s) - Pu(s)\| \\ &\leq \phi(s) + \sup_{w \in C} (\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) + \|u(s) - Pu(s)\| \end{aligned}$$

for all $s, t \in G$ and $\lim_{s \in G} \phi(s) = 0$, where $\phi(s) = \sup_{t \in G} \|u(ts) - S(t)u(s)\|$, we have

$$\|u(ts) - Pu(s)\| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + r - \epsilon = R - \frac{\epsilon}{2}$$

for $s \geq t_0$ and all $t \in G$. Therefore, we obtain

$$\lim_{t \in G} \|u(t) - Pu(s)\| = \inf_{t \in G} \sup_{t \leq \tau} \|u(\tau) - Pu(s)\| \leq R - \frac{\epsilon}{2} < R.$$

This is a contradiction. So we conclude that

$$\inf_{t \in G} \sup_{t \leq s} \|u(s) - Pu(s)\| = R.$$

Now we claim that $\lim_{t \in G} Pu(t) = z_0$. If not, then there exists $\epsilon > 0$ such that for any $t \in G$, $\|Pu(t') - z_0\| \geq \epsilon$ for some $t' \geq t$. Choose $a > 0$ so small that

$$(R + a) \left(1 - \delta \left(\frac{\epsilon}{R + a} \right) \right) = R_1 < R,$$

where δ is the modulus of convexity of the norm of E . We have $\|u(t') - Pu(t')\| \leq R + a$ and $\|u(t') - z_0\| \leq R + a$ for large enough t' . Therefore

$$\left\| u(t') - \frac{Pu(t') + z_0}{2} \right\| \leq (R + a) \left(1 - \delta \left(\frac{\epsilon}{R + a} \right) \right) = R_1.$$

Since, by Lemma 1, the point $w_{t'} = \frac{Pu(t') + z_0}{2}$ belongs to $F(\mathcal{S})$, as in the above,

$$\|u(tt') - w_{t'}\| \leq \phi(t') + \sup_{w \in C} (\|S(t)u(t') - S(t)w\| - \|u(t') - w\|) + \|u(t') - w_{t'}\|.$$

Since $\lim_{s \in G} \phi(s) = 0$, there is $t' \in G$ such that

$$\phi(t') < \frac{R - R_1}{4}$$

and

$$\sup_{t' \preceq t} \sup_{w \in C} (\|S(t)u(t') - S(t)w\| - \|u(t') - w\|) < \frac{R - R_1}{4},$$

and hence

$$\lim_{t \in G} \|u(t) - w_{t'}\| = \inf_{r \in G} \sup_{t \preceq r} \|u(\tau) - w_{t'}\| < \frac{R - R_1}{2} + R_1 = \frac{R + R_1}{2} < R.$$

This contradicts the fact $R = \inf\{g(z) : z \in F(\mathcal{S})\}$. Thus we have $\lim_{t \in G} Pu(t) = z_0$. Consequently, it follows that the element $z_0 \in F(\mathcal{S})$ with $g(z_0) = \min\{g(z) : z \in F(\mathcal{S})\}$ is unique. The proof is complete

By Corollary 1 and Theorem 2, we have the following, which is an improvement of [8, Theorem 3] and [5, Theorem 3.3].

COROLLARY 2. Let C be a nonempty closed convex subset of a real Hilbert space H . Let G be a right reversible semitopological semigroup and $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C . Suppose that $F(\mathcal{S}) \neq \emptyset$. Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Then $\{u(t) : t \in G\}$ converges weakly to some $z \in C$ if and only if $u(ht) - u(t)$ converges weakly to 0 for all $h \in G$. In this case, $z \in F(\mathcal{S})$ and $\lim_{t \in G} Pu(t) = z$.

PROOF. We need only prove the "if" part. By Corollary 1, it suffices to show that $\omega(u) \subset F(\mathcal{S})$. Let $\{u(t_\alpha)\}$ be a subnet of $\{u(t) : t \in G\}$ converging weakly to $y \in C$. Given $\epsilon > 0$. Since \mathcal{S} is of asymptotically nonexpansive type and $\{u(t_\alpha)\}$ is bounded, there exists $t_0 \in G$ such that for any α ,

$$\sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)u(t_\alpha) - S(t)w\| - \|u(t_\alpha) - w\|) < \epsilon.$$

So we have, for $t \succeq t_0$ and any α ,

$$\begin{aligned} & \|S(t)u(t_\alpha) - S(t)y\|^2 - \|u(t_\alpha) - y\|^2 \\ &= (\|S(t)u(t_\alpha) - S(t)y\| - \|u(t_\alpha) - y\|)(\|S(t)u(t_\alpha) - S(t)y\| + \|u(t_\alpha) - y\|) \\ &\leq \sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)u(t_\alpha) - S(t)w\| - \|u(t_\alpha) - w\|) \left(\sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)u(t_\alpha) - S(t)w\| \right. \\ &\quad \left. - \|u(t_\alpha) - w\|) + 2\|u(t_\alpha) - y\| \right) \\ &< \epsilon(\epsilon + 2M), \end{aligned}$$

where $M = \sup_\alpha \|u(t_\alpha) - y\|$. Let $u \in F(\mathcal{S})$ and $\epsilon' = \epsilon(\epsilon + 2M)$. Then we have, for $t \succeq t_0$ and all α ,

$$\begin{aligned} -\epsilon' &< \|u(t_\alpha) - y\|^2 - \|S(t)u(t_\alpha) - y\|^2 \\ &= \|u(t_\alpha) - u\|^2 + 2(u(t_\alpha) - u, u - y) + \|u - y\|^2 \\ &\quad - \|S(t)u(t_\alpha) - u\|^2 - 2(S(t)u(t_\alpha) - u, u - S(t)y) - \|u - S(t)y\|^2 \\ &= \|u(t_\alpha) - u\|^2 - \|S(t)u(t_\alpha) - u\|^2 + \|u - y\|^2 - \|u - S(t)y\|^2 \\ &\quad + 2(u(t_\alpha) - u, S(t)y - y) + 2(u(t_\alpha) - S(t)u(t_\alpha), u - S(t)y). \end{aligned}$$

Since $\{u(t) : t \in G\}$ is an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$ and $u(ht) - u(t)$ converges weakly to 0 for all $h \in G$, it follows that

$$\lim_{\alpha} \|S(t)u(t_\alpha) - u\|^2 = \lim_{\alpha} \|u(tt_\alpha) - u\|^2 = \lim_{\alpha} \|u(t_\alpha) - u\|^2$$

$$u(t_\alpha) - S(t)u(t_\alpha) = u(t_\alpha) - u(tt_\alpha) \rightarrow 0 \text{ weakly.}$$

Thus we have

$$-\epsilon' \leq 2(y - u, S(t)y - y) + \|y - u\|^2 - \|u - S(t)y\|^2 = -\|y - S(t)y\|^2$$

for $t \geq t_0$, and hence $\limsup_{t \in G} \|S(t)y - y\| \leq \epsilon'$. Since ϵ' is arbitrary, we have $\lim_{t \in G} S(t)y = y$. Now, for $s \in G$,

$$S(s)y = \lim_{t \in G} S(s)S(t)y = \lim_{t \in G} S(st)y = \lim_{t \in G} S(t)y = y,$$

i.e., $y \in F(S)$ and hence $\omega(u) \subset F(S)$. By Corollary 1, the net $\{u(t) : t \in G\}$ converges weakly to some $z \in F(S)$. On the other hand, since P is the metric projection of H onto $F(S)$, we know that

$$(u(t) - Pu(t), Pu(t) - y) \geq 0$$

for all $y \in F(S)$. So, if $Pu(t) \rightarrow u$ by Theorem 2, we have $(z - u, u - y) \geq 0$ for all $y \in F(S)$. Putting $z = y$, we obtain $-\|z - u\|^2 \geq 0$ and hence $z = u$.

As a direct consequence, we have the following:

COROLLARY 3. Let C be a nonempty closed convex subset of a real Hilbert space H . Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C . Suppose that $F(\mathcal{S}) \neq \emptyset$. Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. If $\lim_{t \in G} \|u(ht) - u(t)\| = 0$ for all $h \in G$, then the net $\{u(t) : t \in G\}$ converges weakly to some $z \in F(\mathcal{S})$.

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